NOTES FOR COURSE ON F-SINGULARITIES

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1. Basics and notation

Unless otherwise specified, all rings involved are commutative Noetherian with identity, and all ring homomorphisms are unitary.

Recall that a local ring $(R, \mathfrak{m})$ ring is said to have *equal characteristic* (or to be equi-characteristic) if $\mathrm{char}(R) = \mathrm{char}(R/\mathfrak{m})$. In these notes, most of the time we will consider rings of equal characteristic $p > 0$ where p is, of course, a prime integer.

Remark 1.1. Since $\mathrm{char}(R/\mathfrak{m})$ always divides $\mathrm{char}(R)$, if the latter is a prime, then R is necessarily equi-characteristic. In fact, if R is not necessarily local and has prime characteristic $p > 0$, then every localization of R must have equal characteristic p. Moreover, the condition that $\mathrm{char}(R)$ is a prime p is equivalent to the fact that R contains a field of characteristic p. In fact, $\mathrm{char}(R) = p$ means that the natural map $\mathbb{Z} \to R$ sending $1_{\mathbb{Z}} \to 1_R$ has kernel $p\mathbb{Z}$. Therefore R contains the finite field $\mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$. Conversely, if R contains a field ℓ of characteristic $p > 0$, then $\mathrm{char}(R) = p$ is readily seen to be forced.

1.1. The Frobenius map.

Let R be a ring of prime characteristic $p > 0$. The *Frobenius endomorphism* on R is the map

$$\begin{aligned} F : R &\longrightarrow R \\ r &\longrightarrow r^p \end{aligned}$$

Lemma 1.2. *The Frobenius map is a ring homomorphism.*

Proof. Let $r, s \in R$. Since R is commutative, it is clear that $F(rs) = (rs)^p = r^p s^p = F(r)F(s)$. The key point is that it is additive:

$$F(r+s) = (r+s)^p = \sum_{i=0}^{p} \binom{p}{i} r^i s^{p-i} = r^p + s^p = F(r) + F(s),$$

where the third equality follows from the fact that the integer $\binom{p}{i}$ is divisible by p for all $0 < i < p$, and $p = 0$ in R. □

Here are some examples.

Examples 1.3.
1. Let $R = \mathbb{F}_p$ be the field with p elements. Then F is the identity. In particular, it is an isomorphism.
2. Let $R = \mathbb{F}_p[t]$, or $R = \mathbb{F}_p(t) = \mathrm{Frac}(\mathbb{F}_p[t])$. Then F is injective but not surjective; for instance, t is not in the image. We say that a ring is *perfect* if F is a surjective homomorphism.

Exercise 1.4. Prove that, if R is a Noetherian perfect ring, then R is a direct product of fields.

Regarding injectivity, instead, the following is easy to prove.

Proposition 1.5. *Let R be a ring of prime characteristic $p > 0$. Then R is reduced if and only if $F : R \to R$ is injective.*

Proof. Assume R is reduced. If $F(r) = r^p = 0$, then $r = 0$. So F is injective. Conversely, assume that F is injective. Let $r \in R$ and assume that $r^N = 0$ for some $N \in \mathbb{N}^*$. Let $e = \inf\{e' \in \mathbb{N} \mid r^{p^{e'}} = 0\}$, which is finite by assumption. If $e \geqslant 1$, then set $s = r^{p^{e-1}}$. By definition of e, we have $s \neq 0$, but $F(s) = r^{p^e} = 0$, contradicting our assumptions. Therefore $e = 0$, which means that $r^{p^0} = r = 0$. This completes the proof. $\square$

We now define Frobenius powers of an ideal $I \subseteq R$:

$$I^{[p]} = (x^p \mid x \in I).$$

Observe that $I^{[p]} = F(I)R$, where F is the Frobenius map. For this reason, if $I = (f_1, \ldots, f_t)$, one has $I^{[p]} = (f_1^p, \ldots, f_t^p)$.

Exercise 1.6. Assume that R contains $\mathbb{Q}$. Let $I \subseteq R$ be an ideal, and let $n \in \mathbb{N}^*$. Define $I^{[n]} = (x^n \mid x \in I)$. Prove that $I^{[n]} = I^n$, the ordinary power of I.

Notation 1.7. For a ring R of prime characteristic $p > 0$, we use $q, q', q'', \ldots$ to denote powers $p^e, p^{e'}, p^{e''}, \ldots$ of p, for $e, e', e'' \in \mathbb{N}$. For an ideal I, we then write $I^{[q]}$ for $I^{[p^e]}$, etc.

1.2. **Frobenius push-forward.** Given any ring homomorphism $f : R \to S$, one can view S as an R-module by restriction of scalars. In other words, given $r \in R$ and $s \in S$, the action $r \cdot s = f(r)s$ makes S into an R-module.

In the case of the Frobenius map $F : R \to R$, the action that makes R into an R-module (via F) can be confused with the standard action. For this reason, it is often convenient to use different notations for R as a source and as a target of F. If we let $F_*(R)$ denote R when viewed as a module over itself via Frobenius, and for $r \in R$ we denote by $F_*(r)$ its elements (just to distinguish them from the scalars), then for $r, s \in R$ we have

$$F_*(r) + F_*(s) = F_*(r + s) \quad \text{and} \quad r \cdot F_*(s) = F_*(r^p s).$$

Observe that the natural map $\varphi : R \to F_*(R)$ which sends $r \mapsto F_*(r^p)$ is now R-linear. In fact, for $r, s \in R$ we have

$$\varphi(r + s) = F_*((r + s)^p) = F_*(r^p + s^p) = F_*(r^p) + F_*(s^p) = \varphi(r) + \varphi(s),$$

and

$$r\varphi(s) = rF_*(s^p) = F_*(r^p s^p) = F_*((rs)^p) = \varphi(rs).$$

Remark 1.8. The same considerations can be carried out for the e-th iteration of Frobenius. In this case, we will denote the restriction of scalars by $F_*^e(R)$, and the map $R \to F_*^e(R)$ sending $r \mapsto F_*^e(r^{p^e})$ is now R-linear.

Now let M be any R-module. We let $F_*(M)$ be the $F_*(R)$-module, with operations defined as follows. For $m_1, m_2 \in M$ and $r \in R$:

$$F_*(m_1) + F_*(m_2) = F_*(m_1 + m_2) \quad \text{and} \quad F_*(r) \cdot F_*(m_1) = F_*(rm_1).$$

This action is not very interesting: in fact, if we recall that R and $F_*(R)$ are actually the same ring, and we identify them, then M and $F_*(M)$ become the same module.

However, every $F_*(R)$-module is also an R-module. In this case, the action is much more interesting: $r \cdot F_*(m) = F_*(r^p m)$ for all $r \in R$ and $m \in M$. We will use this action over and over again.

Given an R-linear map $f : M \to N$, there is an induced $F_*(R)$-linear map $F_*(f) : F_*(M) \to F_*(N)$, defined as $F_*(f)(F_*(m)) = F_*(f(m))$ for all $m \in M$. In particular, this is also an R-linear map.

Remark 1.9. An alternative point of view, which is only fully justified when R is reduced, is that of viewing $F_*(R)$ as the ring of p-th roots of elements in R. More generally, given an R-module M, we let $M^{1/p}$ be the set of elements $m^{1/p}$, for $m \in M$. Under this point of view we have that the map $R \to R^{1/p}$ sending $r \mapsto r = (r^p)^{1/p}$ is R-linear, and every $M^{1/p}$ is an $R^{1/p}$-module, as well as an R-module.

Proposition 1.10. *Let R be a ring of characteristic $p > 0$, and e be a positive integer.*

(1) *The functor $F_*^e(-)$ from the category of R-modules to the category of $F_*^e(R)$-modules is exact.*
(2) *If $W \subseteq R$ is a multiplicatively closed system, then the R_W-linear map $\psi : (F_*^e(R))_W \to F_*^e(R_W)$ defined as $\frac{F_*^e(r)}{w} \mapsto F_*^e\left(\frac{r}{w^{p^e}}\right)$ is an isomorphism.*

Proof. Part (1) is clear, since if $0 \to A \to B \to C \to 0$ is any exact sequence of R-modules, then both the modules and the maps in the sequence $0 \to F_*^e(A) \to F_*^e(B) \to F_*^e(C) \to 0$ are unchanged (what changes is the R-module structure). In particular, kernels and images are unchanged, and the sequence is still exact.

For (2): the map ψ is clearly additive. Given $s/u \in R_W$, letting $q = p^e$ we have

$$\psi\left(\frac{s}{u} \cdot \frac{F_*^e(r)}{w}\right) = \psi\left(\frac{F_*^e(s^q r)}{uw}\right) = F_*^e\left(\frac{s^q r}{u^q w^q}\right) = \frac{s}{u} F_*^e\left(\frac{r}{w^q}\right) = \frac{s}{u} \cdot \psi\left(\frac{F_*^e(r)}{w}\right).$$

Now consider the map $\varphi : F_*^e(R_W) \to (F_*^e(R))_W$ defined as $F_*^e\left(\frac{r}{w}\right) \mapsto \frac{F_*^e(rw^{q-1})}{w}$. It is clearly additive, and for $s/u \in R_W$ we have

$$\begin{aligned}\varphi\left(\frac{s}{u} \cdot F_*^e\left(\frac{r}{w}\right)\right) = \varphi\left(F_*^e\left(\frac{s^q r}{u^q w}\right)\right) &= \frac{F_*^e(s^q r u^{q(q-1)} w^{q-1})}{u^q w} \\ &= \frac{s u^{q-1} F_*^e(rw^{q-1})}{u^q w} = \frac{s}{u} \cdot \frac{F_*^e(rw^{q-1})}{w} = \frac{s}{u} \cdot \varphi\left(F_*^e\left(\frac{r}{w}\right)\right).\end{aligned}$$

Thus both ψ and φ are R_W-linear. It is easy to check that they are each other's inverse, and therefore ψ is an isomorphism.

□

Remark 1.11. From the point of view of p^e-th roots, the previous proposition simply states the more intuitive fact that the functor $(-)^{1/p^e}$ is exact, and that taking p^e-th roots commutes with localization.

1.3. **A quick reminder of integral closure.**

Definition 1.12. Let $I \subseteq R$ be an ideal. An element $x \in R$ is said to be *integral over I* if there exists $N > 0$ and elements $i_j \in I^j$ such that

$$x^N + i_1 x^{N-1} + \ldots + i_N = 0.$$

We denote by $\overline{I}$ the set of all elements that are integral over I.

The following are some properties that are almost immediate to check. We refer to [HS06] if the reader is interested in seeing a proof.

Proposition 1.13. *Let $I \subseteq R$ be an ideal, and $x \in R$ be an element.*

(1) *$\overline{I}$ is an ideal, and $I \subseteq \overline{I} \subseteq \sqrt{I}$.*

(2) *$x \in \overline{I}$ if and only if the image of x in R/P belongs to $\overline{IR/P}$ for every minimal prime P of R.*
(3) *If $R_{red} = R/\sqrt{0}$, then $\overline{I}R_{red} = \overline{IR_{red}}$.*
(4) *$x \in \overline{I}$ if and only if there exists $N > 0$ such that $x^{N+r} \in I^r$ for all $r \geqslant 1$.*
(5) *If W is a multiplicatively closed system, then $\overline{I}R_W = \overline{IR_W}$.*

For our purposes, it will be helpful to give a different characterization of integral elements. The full proof of the characterization can be found in [HS06], and relies on the concept of valuation. Here, we will only prove one implication.

Notation 1.14. For a ring R, let us denote by R° the set of elements in R that do not belong to any minimal prime. For example, if R is a domain, then $R^\circ = R \smallsetminus \{0\}$.

Proposition 1.15. *Let $I \subseteq R$ be an ideal, and $x \in R$ be an element. Then $x \in \overline{I}$ if and only if there exists $c \in R^\circ$ such that $cx^n \in I^n$ for infinitely many $n \gg 0$.*

Proof. Let $\operatorname{Min}(R) = \{P_1, \ldots, P_s\}$. First assume that $x \in \overline{I}$. By Proposition 1.13 (4) we have that $x^{N+r} \in I^r(I+(x))^N \subseteq I^r$ for all $r \geqslant 1$. Assume that I is contained in $P_1 \cap \ldots \cap P_k$, and I is not contained in P_j for $j = k+1, \ldots, s$. By prime avoidance we can choose $d \in I^N \smallsetminus (P_{k+1} \cup \ldots \cup P_s)$. Moreover, let $t \gg 0$ be such that $(\sqrt{0})^t = (P_1 \cap \ldots \cap P_s)^t = (0)$, and let $e \in (P_{k+1} \cap \ldots \cap P_s)^t \smallsetminus (P_1 \cup \ldots \cup P_k)$. Observe that, by our choices, $e(P_1 \cap \ldots \cap P_k)^t = (0)$. Let $c = d + e$, and note that $c \in R^\circ$; in fact, if for instance $c \in P_1$, then $e = c - d \in P_1$, because $d \in I^N \subseteq P_1$, and this contradicts our choice of e. On the other hand, if $c \in P_s$, then $d = c - e \in P_s$, contradicting our choice of d. Finally, by what we have shown above, we have that $cx^{N+r} \in cI^r = (d+e)I^r \subseteq I^{N+r} + eI^r$. For $r \geqslant t$ we have that $eI^r \subseteq e(P_1 \cap \ldots \cap P_k)^t = (0)$, and this shows that $cx^n \in I^n$ for all $n \gg 0$.

The converse relies on valuations. Assume that $cx^n \in I^n$ for all $n \gg 0$, and let $P = P_j$ be any minimal prime of R. Let V be a discrete valuation domain sitting between R/P and $\operatorname{Frac}(R/P)$, with associated value function $v : V \to \mathbb{Z}$. Note that $cx^n \in I^nV$ still holds for infinitely many $n \gg 0$, and therefore $v(c) + nv(x) \geqslant nv(I)$. Since this holds for all $n \gg 0$, one must have $v(x) \geqslant v(I)$, that is, $x \in \overline{IV}$. Since $\overline{I} = \bigcap_V \overline{IV} \cap R$, the claim follows. $\square$

1.4. **Tight closure.** Tight closure was introduced by Hochster and Huneke around 1990 as a systematic tool to attack problems for rings of characteristic $p > 0$. Its definition can seem quite technical and obscure at the beginning, but it is very natural, especially if compared with the characterization of integral closure given in Proposition 1.15. We will make this connection later in this subsection.

Definition 1.16. Let R be a ring of prime characteristic $p > 0$, $I \subseteq R$ be an ideal, and $x \in R$ be an element. We say that x belong to the *tight closure of I* if there exists $c \in R^\circ$ such that $cx^q \in I^{[q]}$ for all $q = p^e \gg 0$. We denote by I^* the set of elements in R that belong to the tight closure of I. We say that I is tightly closed if $I = I^*$.

Example 1.17. Let $R = \mathbb{F}_3[x,y]/(x^2 - y^3)$. Then $x \in (y)^*$. In fact, choose $c = x \in R^\circ$; for all $e \in \mathbb{N}^*$ we have

$$x \cdot x^{3^e} = (x^2)^{\frac{3^e+1}{2}} = (y^3)^{\frac{3^e+1}{2}} = y^{3^e} \cdot y^{\frac{3^e+3}{2}} \in (y^{3^e}) = (y)^{[3^e]}.$$

Some basic properties of tight closure:

Proposition 1.18. *Let $I \subseteq R$ be an ideal. Then:*

(1) *I^* is an ideal.*
(2) $I \subseteq I^*$.
(3) *If $I \subseteq J$, then $I^* \subseteq J^*$.*
(4) $(I^*)^* = I^*$.

Proof. (1) If $x, y \in I^*$, then there exist $c, d \in R^\circ$ such that $cx^q \in I^{[q]}$ and $dy^q \in I^{[q]}$ for $q = p^e \gg 0$. Observe that $cd(x+y)^q \in I^{[q]}$ for $q \gg 0$, proving that $x + y \in I^*$. Similarly, if $x \in I^*$ and $r \in R$, then $cx^q \in I^{[q]}$ for $q \gg 0$, and it follows that $c(rx)^q = cr^q x^q \in I^{[q]}$ for $q \gg 0$, so that $rx \in I^*$. (2) and (3) are equally straightforward.

(4) By (2), it suffices to show one containment. Let $x \in (I^*)^*$. Then $cx^q \in (I^*)^{[q]}$ for $q \gg 0$. Let $I^* = (f_1, \ldots, f_t)$. Then, there exist $c_1, \ldots, c_t \in R^\circ$ such that $c_i f_i^q \in I^{[q]}$ for all $q \gg 0$ (and we choose $q \gg 0$ that works for all elements here involved). Let $d = c \cdot c_1 \cdots c_t$, and observe that $d \in R^\circ$. Then for some $r_i \in R$ and all $q \gg 0$ we have

$$dx^q = c_1 \cdots c_t \left(\sum_i r_i f_i^q \right) = \sum_i r_i' c_i f_i^q \in I^{[q]},$$

and it follows that $x \in I^*$. □

Some more examples, to show the subtlety of the definition.

Examples 1.19. (1) Let $R = \mathbb{F}_p[x,y,z]/(x^3+y^3+z^3)$, with $p > 3$. Then $x^2 \in (y,z)^*$.
(2) Let $R = \mathbb{F}_p[x,y,z]/(x^2-y^3-z^5)$. Then $(y,z)^* = (y,z)$. This can be tested by showing that $x \notin (y,z)^*$, because x generates the socle of $R/(y,z)$.
(3) Let $R = \mathbb{F}_p[x,y,z]/(x^2-y^5-z^7)$. This time, $x \in (y,z)^*$.

Proposition 1.18 shows that tight closure is indeed a closure operation.

Proposition 1.20. *Let $I \subseteq R$ be an ideal. Then:*
(1) $\sqrt{0} \subseteq I^* \subseteq \overline{I} \subseteq \sqrt{I}$. *Moreover, if I is a principal ideal, then $I^* = \overline{I}$.*
(2) *$x \in I^*$ if and only if the image of x in R/P belongs to the tight closure of $(I+P)/P$ for every minimal prime P of R.*
(3) *If R is reduced, or I has positive height, then $x \in I^*$ if and only if there exists $c \in R^\circ$ such that $cx^q \in I^{[q]}$ for all q.*

Proof. The inclusions of (1) are easy from the definitions and the previous discussion about $\overline{I}$. For the second claim, it suffices to observe that $I^{[q]} = I^q$ if I is principal.

(2) If $x \in I^*$, then the same relation holds true when going modulo any minimal prime P. Observe that $c \in R^\circ$ implies that the class of c in R/P is non-zero, hence $\overline{c} \in (R/P)^\circ$. For the converse, let $P_1, \ldots, P_s$ be the minimal primes of R, and let x be such that $x_i \in I_i^*$, where the subscript i denotes the class of x and I in R/P_i. By assumption, there exist $c_i \in (R/P_i)^\circ = R/P_i \smallsetminus \{0\}$ such that $c_i x_i^q \in I_i^{[q]}$ for all $q \gg 0$. Lifting back to to R, we can always assume that the lift c_i' is in R°, by Prime Avoidance. Then $c_i' x^q \in I^{[q]} + P_i$ for all $q \gg 0$. For each i, by Prime Avoidance choose t_i in all the minimal primes except P_i. Let $d = \sum t_i c_i$. Then $dx^q \in I^{[q]} + \sum t_i P_i \subseteq I^{[q]} + \prod_i P_i \subseteq I^{[q]} + \sqrt{0}$. Choose q_0 such that $\left(\sqrt{0}\right)^{[q_0]} = 0$. Then $c^{q_0} x^{qq_0} \in I^{[qq_0]}$, and thus $x \in I^*$.

(3) Let I be an ideal, and $x \in I^*$. Fix $c \in R^\circ$ and q_0 such that $cx^q \in I^{[q]}$ for all $q \geqslant q_0$. First assume that $\operatorname{ht}(I) > 0$. Then there exists $d \in I$ that avoids all minimal primes of R. In particular, $d^{q_0} \in R^\circ$, and also $d^{q_0} \in I^{[q]}$ for all $q \leqslant q_0$. Setting $e = cd^{q_0}$, we have

that $ex^q \in I^{[q]}$ for all q. Now assume that R is reduced instead; by what we have already proved, we may assume that $\mathrm{ht}(I) = 0$. In other words, if $(0) = P_1 \cap \ldots \cap P_t$, where the P_i's are the minimal primes of R, then $P_i \in \mathrm{Ass}(R/I)$ for some i (possibly more than one). Assume that $P_1, \ldots, P_s \in \mathrm{Ass}(R/I)$, while $P_{s+1}, \ldots, P_t \notin \mathrm{Ass}(R/I)$. By (1), we have that $x \in I^* \subseteq \sqrt{I} \subseteq P_1 \cap \ldots \cap P_s$, and $x \notin \bigcup_{j=s+1}^t P_j$. By Prime Avoidance, we can find $d \in P_{s+1} \cap \ldots \cap P_t$ such that $d \notin \bigcup_{i=1}^s P_i$. Observe that $xd \in \bigcap_{i=1}^t P_i = (0)$. Again by Prime Avoidance, we can find $\alpha \in I$, with $\alpha \notin \bigcup_{j=s+1}^t P_j$; observe that $\alpha^{q_0} \in I^{[q_0]} \smallsetminus \bigcup_{j=s+1}^t P_j$. Let $e = c\alpha^{q_0} + d$. Note that $e \in R^\circ$ by choice of α and d. Moreover, we have $ex^q = \alpha^{q_0} cx^q \in I^{[q]}$ for all q, as desired. □

Remark 1.21. A different way to express the condition that $x \in I^*$ is using Frobenius push-forwards. To see this better first assume that R is reduced, so we may use the point of view of p-th roots. Then $cx^q \in I^{[q]}$ is equivalent, after taking q-th roots, to the condition that $c^{1/q}x \in (I^{[q]})^{1/q} = IR^{1/q}$. More generally, using Frobenius push-forwards, we have that $x \in I^*$ if and only if there exists $c \in R^\circ$ such that $F^e_*(c)x \in IF^e_*(R)$ for all $e \gg 0$.

2. Kunz's Theorem and tight closure in regular rings

Using the last remark of the previous section (or Proposition 1.18 (2)), we can view the condition of belonging to the tight closure of an ideal as a weakening of the membership condition. In this sense, if an ideal is tightly closed, it is "easier" to show that an element belongs to the tight closure, then to the ideal itself. So it makes sense to ask: *when are ideals tightly closed?*

We start with an illustrative example.

Example 2.1. Let $R = \mathbb{F}_p[\![x, y]\!]$, and $I = (x^2, y^2)$. Observe that $(xy)^2 \in I^2$, therefore $(xy)^{2n} \in I^{2n}$ for all n. This gives $xy \in \overline{I}$. On the other hand, if $xy \in I^*$, we would have $c(xy)^q \in I^{[q]} = (x^{2q}, y^{2q})$ for all $q \gg 0$. Then $c \in (x^{2q}, y^{2q}) :_R (xy)^q = (x^q, y^q)$. This gives $c \in \bigcap_{q \gg 0}(x^q, y^q) \subseteq \bigcap_{q\gg 0} \mathfrak{m}^q = (0)$. A contradiction. So $xy \notin I^*$ and, in fact, $J = J^*$ for all $J \subseteq R$.

Observe that $R = \mathbb{F}_p[\![x, y]\!]$ is a regular (local) ring. More generally, we have the following family of examples.

Example 2.2. Let R be either $\mathbb{F}_p[x_1, \ldots, x_d]$ or $\mathbb{F}_p[\![x_1, \ldots, x_d]\!]$. Since $F^e(R) = R^q \cong \mathbb{F}_p[x_1^q, \ldots, x_d^q]$ in the first case and $\mathbb{F}_p[\![x_1^q, \ldots, x_d^q]\!]$ in the second, it can easily be proved by induction on $d \geqslant 1$ that R is a free R^q-module, with basis given by $\{x_1^{i_1} \cdots x_d^{i_d} \mid 0 \leqslant i_1, \ldots, i_d \leqslant q-1\}$. In particular, the rank of R as an R^q-module is q^d. This is equivalent to claiming that $R^{1/q}$ is a free R-module of rank q^d with basis $\{(x_1^{i_1} \cdots x_d^{i_d})^{1/q} \mid 0 \leqslant i_1, \ldots, i_d \leqslant q-1\}$, or that $F^e_*(R)$ is a free R-module of rank q^d, and with basis $\{F^e_*(x_1^{i_1} \cdots x_d^{i_d}) \mid 0 \leqslant i_1, \ldots, i_d \leqslant q-1\}$.

In Example 2.2 we have in particular that $F^e_*(R)$ is flat as an R-module and, since R is reduced, this is the same as the e-th iteration of the Frobenius map being flat. We now recall the definition and the basic properties of flatness that we will need.

Definition 2.3. Let R be a ring, and M be an R-module. The module M is flat over R if, for every exact sequence $0 \to A \to B$ of R-modules, the sequence $0 \to A \otimes_R M \to B \otimes_R M$ is exact. The module M is faithfully flat over R if, for every sequence $\mathcal{E} : A \to B \to C$ (not even necessarily a complex), we have that $\mathcal{E}$ is exact if and only if $\mathcal{E} \otimes_R M$ is exact.

A ring homomorphism $f : R \to S$ is said to be (faithfully) flat if S is a (faithfully) flat R-module via f.

Clearly faithfully flat modules are flat, but the converse does not hold, in general. The following is a basic result on flatness (stated in our very specific case). We give a prove for completeness.

Proposition 2.4. *Let $f : R \to S$ be a ring homomorphism. Then*

(1) *f is flat if and only if, for all $M \in \operatorname{Spec}(S)$ maximal ideal, and $\mathfrak{m} = M \cap R (= f^{-1}(M))$ maximal ideal in R, the induced map $f : R_{\mathfrak{m}} \to S_M$ is flat.*

(2) *f is faithfully flat if and only if it is flat and, for all R-modules $A \neq 0$, we have that $A \otimes_R S \neq 0$.*

(3) *f is faithfully flat if and only if it is flat and $f(\mathfrak{m})S \neq S$ for every maximal ideal $\mathfrak{m}$ of R.*

(4) *If f is faithfully flat, then it is injective.*

Proof. (1) First, assume that f is flat, and let M be a maximal ideal in S. Let $0 \to A \to B$ be an exact sequence of $R_{\mathfrak{m}}$-modules, with $\mathfrak{m} = M \cap R$. We have that $A \otimes_{R_{\mathfrak{m}}} S_M \cong A \otimes_R S_M \cong (A \otimes_R S) \otimes_S S_M$, and similarly for B. Since f is flat, the sequence $0 \to A \otimes_R S \to B \otimes_R S$ is exact. Finally, localization is flat, therefore $0 \to (A \otimes_R S) \otimes_S S_M \to (B \otimes_R S) \otimes_S S_M$ is exact.

Conversely, assume that S_M is a flat $R_{\mathfrak{m}}$-module for all maximal ideals M of S, with $\mathfrak{m} = M \cap R$. Let $0 \to A \to B$ be an exact sequence of R-modules, and let K be the kernel of $A \otimes_R S \to B \otimes_R S$, which is an S-module. Let M be a maximal ideal of S. Then localizing gives an exact sequence $0 \to K_M \to (A \otimes_R S) \otimes_S S_M \to (B \otimes_R S) \otimes_S S_M$. Using the same isomorphisms as above, this is $0 \to K_M \to A_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} S_M \to B_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} S_M$. As S_M is flat over $R_{\mathfrak{m}}$ by assumption, we must have $K_M = 0$. As this holds for all maximal ideals M in S, this implies $K = 0$.

(2) First, assume that f is faithfully flat. Then f is flat. Moreover, if A is an R-module such that $A \otimes_R S = 0$, then the exactness of the sequence $(0 \to A \to 0) \otimes_R S$ implies that $0 \to A \to 0$ is exact, that is, $A = 0$.

Conversely, let $\mathcal{E} : A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence; if $\mathcal{E}$ is exact then $\mathcal{E} \otimes_R S$ is exact, since f is flat by assumption. On the other hand, assume that $\mathcal{E} \otimes_R S : A \xrightarrow{f_S} B \xrightarrow{g_S} C$ is exact. By right-exactness of tensor products we have that $\operatorname{Im}(g \circ f) \otimes_R S \cong \operatorname{Im}(g_S \circ f_S)$, and the latter is zero by assumption. It follows from our hypotheses that $\operatorname{Im}(g \circ f) = 0$, that is $\mathcal{E}$ is a complex. let $H(\mathcal{E})$ denote its homology. Since f is flat, we have that $H(\mathcal{E}) \otimes_R S \cong H(\mathcal{E} \otimes_R S)$, and the latter is zero by assumption. Again, it follows from our hypotheses that $H(\mathcal{E}) = 0$.

(3) If f is faithfully flat, we only have to show that $f(\mathfrak{m})S \neq S$ for all maximal ideals $\mathfrak{m}$ of R. But this is immediate from (2), since $R/\mathfrak{m} \neq 0$ implies that $R/\mathfrak{m} \otimes_R S \cong S/f(\mathfrak{m})S \neq 0$.

Conversely, thanks to (2) we just need to show that if A is an R-module such that $A \otimes_R S = 0$ implies that $A = 0$. By way of contradiction, assume that $A \neq 0$, and let $a \in A$ be a non-zero element. Then $I = \operatorname{ann}_R(a)$ is a proper ideal of R, and thus it is contained in some maximal ideal $\mathfrak{m}$. Note that $aR \cong R/\operatorname{ann}_R(a) = R/I$ Since f is flat, the inclusion $0 \to aR \to A$ gives an inclusion $0 \to aR \otimes_R S \to A \otimes_R S$, and since the latter is zero we have that $0 = aR \otimes_R S \cong R/I \otimes_R S = S/f(I)S$, that is $f(I)S = S$. But then $S = f(I)S \subseteq f(\mathfrak{m})S$ implies that $f(\mathfrak{m})S = S$, a contradiction. Therefore $A = 0$.

(4) Let $x \in R$ be such that $f(x) = 0$. By flatness we have that $xR \otimes_R S \cong f(x)S = 0$. By (2) we have that $xR = 0$, that is, $x = 0$. $\square$

Observe that, when f is the Frobenius map, Proposition 2.4 shows that $F : R \to R$ is flat if and only if $F : R_\mathfrak{m} \to R_\mathfrak{m}$ is flat for all maximal ideals $\mathfrak{m}$ of R. Moreover, when $(R, \mathfrak{m})$ is local, the condition that $F(\mathfrak{m})R \neq R$ is trivially satisfied. Therefore $F : R \to R$ is flat if and only if $F : R_\mathfrak{m} \to R_\mathfrak{m}$ is faithfully flat for all maximal ideals $\mathfrak{m}$ of R.

Before diving into the proof of Kunz's Theorem, we need some results due to Lech, which can be found in [Lec64].

Definition 2.5. Let $(R, \mathfrak{m})$ be a local ring. A collection of elements $x_1, \ldots, x_n$ is said to be *Lech-independent* if any combination $a_1x_1 + \ldots + a_nx_n = 0$ implies that $a_i \in (x_1, \ldots, x_n)$ for every $i = 1, \ldots, n$.

Equivalently, if we set $\mathfrak{q} = (x_1, \ldots, x_n)$, then $x_1, \ldots, x_n$ are Lech-independent if and only they minimally generate $\mathfrak{q}$, and $\mathfrak{q}/\mathfrak{q}^2$ is a free $R/\mathfrak{q}$-module.

Lemma 2.6 (Lech's Lemma)**.** *Let $(R, \mathfrak{m})$ be a local ring, and $x_1, \ldots, x_n$ be Lech-independent elements which generate an $\mathfrak{m}$-primary ideal. If $x_1 = y_1z_1$, then*

$$\ell_R(R/(x_1, x_2, \ldots, x_n)) = \ell_R(R/(y_1, x_2, \ldots, x_n)) + \ell_R(R/(z_1, x_2, \ldots, x_n)).$$

Proof. We prove that $(x_1, \ldots, x_n) :_R y_1 = (z_1, x_2, \ldots, x_n)$. The containment $\supseteq$ is clear. Conversely, if $ay_1 + a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$, then multiplying by z_1 and rearranging we get that $(a + a_1z_1)x_1 + b_2x_2 + \ldots + b_nx_n = 0$, where $b_i = z_1a_i$. Our assumption yields that $a + a_1z_1 \in (x_1, \ldots, x_n) \subseteq (z_1, x_2, \ldots, x_n)$, and therefore $a \in (z_1, x_2, \ldots, x_n)$, as desired.

To conclude the proof it suffices to consider the short exact sequence

$$0 \longrightarrow R/(x_1, \ldots, x_n) :_R y_1 \xrightarrow{\cdot y_1} R/(x_1, \ldots, x_n) \longrightarrow R/(y_1, x_2, \ldots, x_n) \longrightarrow 0$$

and to count lengths. $\square$

We are now ready to prove Kunz's Theorem [Kun69].

Theorem 2.7 (Kunz)**.** *A ring R is regular if and only if the Frobenius map is flat.*

Proof. Since both issues are local, we may assume that $(R, \mathfrak{m})$ is local. Observe that If R is regular, so is its completion $\widehat{R}$; in fact, by Cohen's structure Theorem, if $k = R/\mathfrak{m}$ then $\widehat{R}$ is isomorphic to a power series ring $k[\![x_1, \ldots, x_d]\!]$. Observe that $\widehat{R}^p \cong k^p[\![x_1^p, \ldots, x_d^p]\!]$. It is easy to see that $k^p[\![x_1, \ldots, x_d]\!]$ is a free $k^p[\![x_1^p, \ldots, x_d^p]\!]$-module; in particular, $k^p[\![x_1^p, \ldots, x_d^p]\!] \to k^p[\![x_1, \ldots, x_d]\!]$ is flat. Since $k^p[\![x_1, \ldots, x_d]\!] \to k[\![x_1, \ldots, x_d]\!]$ is also flat, it follows that $\widehat{R}$ is a flat $\widehat{R}^p$-module. Since $\widehat{R}$ is faithfully flat over R, it follows that R is flat over R^p, and since R is reduced this is equivalent to $F : R \to R$ being flat, because the Frobenius map is injective.

For the converse, if the Frobenius map F is flat, then it is in fact faithfully flat, because it is a flat local morphism. Also, the same is true for any iteration of F. Then Frobenius is injective, R is reduced, and R is flat over R^q for any $q = p^e$. Let $\mathfrak{n}_q = \mathfrak{m} \cap R^q$, and observe that it is the maximal ideal of R^q. Moreover, observe that $\mathfrak{m}^{[q]} = \mathfrak{n}_qR$. Since R is flat over R^q, we have that $\mathfrak{m}^{[q]} = \mathfrak{n}_q \otimes_{R^q} R$, and therefore, again by flatness

$$\begin{aligned}\mathfrak{m}^{[q]}/(\mathfrak{m}^{[q]})^2 &\cong \mathfrak{n}_q/\mathfrak{n}_q^2 \otimes_{R^q} R \cong \left(\mathfrak{n}_q/\mathfrak{n}_q^2 \otimes_{R^q/\mathfrak{n}_q} R^q/\mathfrak{n}_q\right) \otimes_{R^q} R \cong \\ &\cong \mathfrak{n}_q/\mathfrak{n}_q^2 \otimes_{R^q/\mathfrak{n}_q} (R^q/\mathfrak{n}_q \otimes_{R^q} R) \cong \mathfrak{n}_q/\mathfrak{n}_q^2 \otimes_{R^q/\mathfrak{n}_q} R/\mathfrak{n}_qR \cong \mathfrak{n}_q/\mathfrak{n}_q^2 \otimes_{R^q/\mathfrak{n}_q} R/\mathfrak{m}^{[q]}\end{aligned}$$

is a free $R/\mathfrak{m}^{[q]}$-module, since $\mathfrak{n}_q/\mathfrak{n}_q^2$ is a free $R^q/\mathfrak{n}_q$-module, given that $R^q/\mathfrak{n}_q$ is a field. Thus, if $x_1, \ldots, x_d$ are a minimal generating set of $\mathfrak{m}$, then $x_1^q, \ldots, x_d^q$ are Lech-independent. Passing to the completion does not affect lengths, therefore a repeated application of Lemma 2.6 gives that

$$\ell_R\left(R/\mathfrak{m}^{[q]}\right) = \ell_{\widehat{R}}\left(\widehat{R}/(x_1^q, \ldots, x_d^q)\widehat{R}\right) = q^d$$

for all $q = p^e$. By Cohen's Structure Theorem we have that $\widehat{R} \cong k[\![X_1, \ldots, X_d]\!]/I$ for some ideal I. If $I \neq (0)$, we can pick $q' = q^{e'} \gg 0$ such that $I \not\subseteq (X_1^{q'}, \ldots, X_d^{q'})$. Then

$$\begin{aligned}(q')^d &= \ell(\widehat{R}/(x_1^{q'}, \ldots, x_d^{q'})\widehat{R}) \\ &= \ell\left(k[\![X_1, \ldots, X_d]\!]/(I, X_1^{q'}, \ldots, X_d^{q'})\right) \\ &< \ell\left(k[\![X_1, \ldots, X_d]\!]/(X_1^{q'}, \ldots, X_d^{q'})\right) = (q')^d,\end{aligned}$$

which contradicts our previous claim. Therefore $I = (0)$, and $\widehat{R}$ is regular. It follows that R is regular as well. □

We end this section relating regular rings and tight closure. We first need to recall some very-well known properties of flat maps.

Proposition 2.8. *Let $f : R \to S$ be a flat ring homomorphism. Let I, J be ideals of R, and $x \in R$ be an element. The following hold:*

(1) *$f(I :_R x)S = f(I)S :_S f(x)$.*
(2) *$f(I \cap J)S = f(I)S \cap f(J)S$.*
(3) *If f is faithfully flat, then $f(I)S \cap R = I$*

Proof. (1) We have a short exact sequence of R-modules:

$$0 \to \frac{R}{I :_R x} \xrightarrow{\alpha} \frac{R}{I} \to \frac{R}{(I, x)} \to 0$$

where the map α sends the class of an element $r \in R$ to the class of $rx \in R/I$. Observe that it is well defined, since $x(I :_R x) \subseteq I$. Since S is flat over R, this induces an exact sequence

$$0 \to \frac{R}{I :_R x} \otimes_R S \xrightarrow{\alpha \otimes_R \mathrm{id}_S} \frac{R}{I} \otimes_R S \to \frac{R}{(I, x)} \otimes_R S \to 0$$

Using the isomorphism $R/J \otimes_R S \cong S/f(J)S$, this becomes

$$0 \to \frac{S}{f(I :_R x)S} \xrightarrow{\beta} \frac{S}{f(I)S} \to \frac{S}{f(I, x)S} \to 0$$

where β sends the class of an element $s \in S$ to the class of $sf(x)$ in S/IS. This sequence shows that $f(I :_R x)S = \{s \in S \mid sf(x) \in f(I)S\} = f(I)S :_S f(x)$.

(2) The proof is similar to that of (2), and it is left as an exercise.

(3) It follows from the fact that the map $R/I \to R/I \otimes_R S \cong S/IS$ is injective. □

Remark 2.9. In the case of the Frobenius map in a regular ring R, then (1) says that for all $I \subseteq R$, $x \in R$ and $q = p^e$ we have $I^{[q]} :_R x^q = (I :_R x)^{[q]}$. Similarly, (2) states that for all $I, J \subseteq R$, and $q = p^e$, we have $(I \cap J)^{[q]} = I^{[q]} \cap J^{[q]}$.

We can finally state and prove Hochster and Huneke's Theorem on tight closure in regular rings.

Theorem 2.10 (Hochster–Huneke)**.** *Let R be a regular ring. Then every ideal of R is tightly closed.*

Proof. Let $I \subseteq R$ be an ideal. Let $x \in R$ be such that $cx^{p^e} \in I^{[p^e]}$ for all $e \geqslant e_0$, for some $e_0 \in \mathbb{N}^*$ and $c \in R^\circ$. By Proposition 2.8, we then have

$$c \in \bigcap_{e \geqslant e_0} I^{[p^e]} :_R x^{p^e} = \bigcap_{e \geqslant e_0} (I :_R x)^{[p^e]} \subseteq \bigcap_{e \geqslant e_0} (I :_R x)^{p^e}.$$

Assume, by way of contradiction, that $x \notin I$. Then $(I :_R x)$ is a proper ideal, hence contained in some maximal ideal $\mathfrak{m}$. Thus we have $(I :_R x)^{p^e} \subseteq \mathfrak{m}^{p^e}$ for all e. It follows that $c \in \bigcap_{e \geqslant e_0} \mathfrak{m}^{p^e}$, and therefore $c = 0$ in $R_\mathfrak{m}$ by Krull's Intersection Theorem. In particular, c is a zero-divisor, and therefore c belongs to some associated prime of R. Since R is regular, it has no embedded primes, and therefore c belongs to some minimal prime of R. This contradicts our choice of $c \in R^\circ$, and it then follows that $x \in I$. □

3. Weakly F-regular rings

Definition 3.1. A ring R is said to be *weakly F-regular* if every ideal $I \subseteq R$ is tightly closed. R is said to be *F-regular* if, for every multiplicatively closed set $W \subseteq R$, the ring R_W is weakly F-regular.

By Theorem 2.10 we have that regular rings are F-regular. Moreover, F-regular rings are clearly weakly F-regular. Some cases in which the converse to the latter holds have been proved but, in general, it is unknown whether the two notions coincide. The difficulty is of course related to the problem of whether tight closure localizes, that is, whether $I^*R_W = (IR_W)^*$ for every multiplicatively closed set W and every ideal $I \subseteq R$. This is known to be false thanks to the following example provided by Brenner and Monsky in [BM10]. However, we point out that this example does not give any information on the relation between weakly F-regular and F-regular rings.

Example 3.2 (Brenner–Monsky)**.** Let $R = \overline{\mathbb{F}}_2[x, y, z, t]/(z^4 + xyz^2 + x^3z + y^3z + tx^2y^2)$, $I = (x^4, y^4, z^4)$, $W = \overline{\mathbb{F}}_2[t]\setminus\{0\}$ and $f = y^3z^3$, then $f \in (IR_W)^*$, but $f \notin I^*R_W$. In particular, we have $I^*R_W \neq (IR_W)^*$. Hence, tight closure does not commute with localization.

Tight closure is known to behave well with respect to localization in certain cases. This allows us to reduce any weakly F-regularity issue to the local case.

Lemma 3.3. *Let $I \subseteq R$ be an ideal.*

(1) *If I is an ideal of R that is primary to a maximal ideal $\mathfrak{m}$, then $(IR_\mathfrak{m})^* = I^*R_\mathfrak{m}$.*
(2) *If every ideal primary to a maximal ideal is tightly closed, then every ideal of R is tightly closed.*

Proof. (1) First the easy containment: let $x \in I^*$; we want to show that its image in $R_\mathfrak{m}$ belongs to $(IR_\mathfrak{m})^*$ (this is true for any ideal, not necessarily $\mathfrak{m}$-primary). By assumption, there is $c \in R^\circ$ such that $cx^q \in I^{[q]}$ for all $q \gg 0$. It follows that the image of cx^q in $R_\mathfrak{m}$ belongs to $I^{[q]}R_\mathfrak{m} = (IR_\mathfrak{m})^{[q]}$ for all $q \gg 0$. As $c \in R^\circ$, we also have that $c \in (R_\mathfrak{m})^\circ$, and this proves the containment.

For the other containment, let $\alpha = x/y \in R_\mathfrak{m}$ (with $x \in R$ and $y \notin \mathfrak{m}$) be such that $c'\alpha^q \in (IR_\mathfrak{m})^{[q]}$ for some $c' = c/c_2 \in (R_\mathfrak{m})^\circ$. Multiplying by $c_2 y^q$ we get that $\frac{cx^q}{1} \in (IR_\mathfrak{m})^{[q]}$ for all $q \gg 0$. Let $P_1, \ldots, P_t$ be the minimal primes of R, and assume that $c \in P_1 \cap \ldots \cap P_s$, but $c \notin P_j$ for all $j = s+1, \ldots, t$. Since $c \in (R_\mathfrak{m})^\circ$, we must necessarily have $P_i \not\subseteq \mathfrak{m}$ for all $i = 1, \ldots, s$. Let $d \in P_{s+1} \cap \ldots \cap P_t \smallsetminus \bigcup_{j=1}^s P_j$. Then d is nilpotent in $R_\mathfrak{m}$, that is $d^N = 0$ in $R_\mathfrak{m}$ for some (fixed) N. Let $I = (f_1, \ldots, f_r)$. Then we have a relation

$$\frac{(c+d^N)x^q}{1} = \frac{cx^q}{1} = \sum_i \frac{a_{i,q}}{b_{i,q}} f_i^q$$

for some $b_{i,q} \notin \mathfrak{m}$ (depending on q!!). Let $e = c + d^N \in R^\circ$, and $b_q = \prod_i b_{i,q}$. This gives $b_q e x^q \in I^{[q]}$ inside R. Since I is $\mathfrak{m}$-primary, so is $I^{[q]}$ for all q. In particular, since $b_q \notin \mathfrak{m}$, we have $ex^q \in I^{[q]}$ for all $q \gg 0$. This gives $x \in I^*$, and thus $\alpha = x/y \in I^* R_\mathfrak{m}$.

(2) Observe that, given any ideal $I \subseteq R$, we have $I = \bigcap_{\mathfrak{m} \in \mathrm{Max}(R)} \bigcap_{n \in \mathbb{N}} (I + \mathfrak{m}^n)$. Since every ideal $I + \mathfrak{m}^n$ is either R or $\mathfrak{m}$-primary, and is therefore assumed to be tightly closed, we have

$$I^* = \left(\bigcap_{\mathfrak{m},n} (I + \mathfrak{m}^n) \right)^* \subseteq \bigcap_{\mathfrak{m},n} (I+\mathfrak{m}^n)^* = \bigcap_{\mathfrak{m},n} (I + \mathfrak{m}^n) = I. \qquad \square$$

Lemma 3.4. *A ring R is weakly F-regular if and only if $R_\mathfrak{m}$ is weakly F-regular for all maximal ideal $\mathfrak{m}$ of R.*

Proof. Assume that R is weakly F-regular, and let $\mathfrak{m}$ be a maximal ideal. By Lemma 3.3, to check that ideals in $R_\mathfrak{m}$ are tightly closed, it suffices to show $\mathfrak{m}R_\mathfrak{m}$-primary ideals. So let $IR_\mathfrak{m}$ be one such ideal, for some ideal $I \subseteq R$. Again by Lemma 3.3 we have $(IR_\mathfrak{m})^* = I^* R_\mathfrak{m} = IR_\mathfrak{m}$, as desired.

Conversely, assume that $R_\mathfrak{m}$ is weakly F-regular for all maximal ideals $\mathfrak{m}$ in R. Again by Lemma 3.3, to show that R is weakly F-regular it is sufficient to show that $I = I^*$ for all ideals primary to a maximal ideal. So let I be an $\mathfrak{m}$-primary ideal, and $x \in I^*$. Then $\frac{x}{1} \in I^* R_\mathfrak{m} = (IR_\mathfrak{m})^* = IR_\mathfrak{m}$. It follows that $yx \in I$ for some $y \notin \mathfrak{m}$; since I is $\mathfrak{m}$-primary, it follows that $x \in I$, as desired. $\square$

As motivation for introducing weakly F-regular rings, we will end this section by proving that weakly F-regular rings are Cohen-Macaulay and normal (i.e., a ring that locally at every maximal ideal is a domain which is integrally closed in its field of fractions). Since all these issues are local, we will assume for the rest of the section that $(R, \mathfrak{m})$ is local.

Proposition 3.5. *Let $(R, \mathfrak{m})$ be a weakly F-regular local ring. Then R is a normal domain.*

Proof. First of all, R is reduced. In fact, by Proposition 1.20 we have that $\sqrt{0} \subseteq (0)^* = (0)$. Now we prove that R is a domain. Let $(0) = P_1 \cap P_2 \cap \ldots \cap P_t$ be the minimal primes of R, and assume that $t \geqslant 2$. We want to reach a contradiction, which will end the proof. If $t \geqslant 2$, we can pick $x \in P_1 \smallsetminus \bigcup_{i=2}^t P_i$ and $y \in P_2 \cap \ldots \cap P_t \smallsetminus P_1$, by Prime Avoidance. Observe that $xy = 0$. For this reason, for all $q = p^e$ we have $(x+y)x^q = x(x+y)^q \in (x+y)^{[q]}$. Observe that $x + y \in R^\circ$, by choice of x and y. Therefore $x \in (x+y)^* = (x+y)$. This says that there exists $r \in R$ such that $x = r(x+y)$, that is, $(1-r)x = ry$. If $r \notin \mathfrak{m}$, then $y \in (x) \in P_1$, a contradiction. If $r \in \mathfrak{m}$, then $1 - r \notin \mathfrak{m}$, so that $x \in (y) \in P_2$, a contradiction again.

Finally, let $\alpha = \frac{r}{s} \in \mathrm{Frac}(R)$ be an element which is integral over R. Then, there exist elements $t_1, \ldots, t_N \in R$ such that $\alpha^N + t_1\alpha^{N-1} + \ldots + t_N = 0$. Multiplying the equation by s^N gives an equation $r^N + t_1 s r^{N-1} + \ldots + t_N s^N = 0$ in R. Note that this gives $r \in \overline{(s)}$. By Proposition 1.15, we can find $c \in R^\circ$ such that $cr^n \in (s)^n = (s^n)$ for all $n \gg 0$. In particular, $cr^q \in (s^q) = (s)^{[q]}$ for all $q = p^e \gg 0$, and thus $r \in (s)^*$. Since R is weakly F-regular, $r \in (s)$, and therefore there is $t \in R$ such that $r = ts$. It follows that $\alpha = \frac{r}{s} = t \in R$, and R is normal. $\square$

We recall that a parameter for a ring $(R, \mathfrak{m})$ of positive dimension is an element $x \in \mathfrak{m}$ such that $\dim(R/(x)) = \dim(R) - 1$. A system of parameters is a sequence of elements $x_1, \ldots, x_d$ such that x_{t+1} is a parameter for $R/(x_1, \ldots, x_t)$. We recall that R is said to be *Cohen-Macaulay* if every system of parameters $x_1, \ldots, x_d$ satisfies $(x_1, \ldots, x_t) :_R x_{t+1} = (x_1, \ldots, x_t)$ for every t.

Theorem 3.6 (Colon capturing)**.** *Let $(R, \mathfrak{m})$ be a local ring that is the homomorphic image of a Cohen-Macaulay local ring. Let $x_1, \ldots, x_d$ be system of parameters. Then*

$$(x_1, \ldots, x_t) :_R x_{t+1} \subseteq (x_1, \ldots, x_t)^*.$$

Proof. Let $P \in \mathrm{Min}(R)$. By Proposition 1.20 (2) it suffices to show the containment in R/P; therefore we may assume without loss of generality that R is a domain. By assumption $R = S/Q$, where Q is a prime of S, say of height h. Let $y_1, \ldots, y_h \in Q$ be a regular sequence and choose $z_1' \in S$ be any lift of x_1. By assumption, z_1' does not belong to Q. Since Q is a minimal prime over $J = (y_1, \ldots, y_h)$ and S is Cohen-Macaulay, we have that $\mathrm{Ass}(S/J) = \mathrm{Min}(J) = \{Q, Q_1, \ldots, Q_s\}$, where $Q_1, \ldots, Q_s$ are primes of the same height h. Assume that $z_1' \in Q_1 \cap \ldots \cap Q_t$, and $z_1' \notin Q_{t+1} \cup \ldots \cup Q_s$. By Prime Avoidance, we can choose $z_1'' \in Q \cap Q_{t+1} \cap \ldots \cap Q_s \smallsetminus (Q_1 \cup \ldots \cup Q_t)$. Observe that $z_1 = z_1' + z_1''$ avoids all associated primes of $(y_1, \ldots, y_h)$, and it is still a lift of x_1, since $z_1'' \in Q$. Therefore $y_1, \ldots, y_h, z_1$ forms a regular sequence. Repeating this argument with a lift z_t' of x_t and the ideal $Q + (z_1, \ldots, z_{t-1})$ we obtain elements $z_1, \ldots, z_d$ of S such that $(z_1, \ldots, z_t)S/Q = (x_1, \ldots, x_t)S/Q$ for all $t = 1, \ldots, d$, and $y_1, \ldots, y_h, z_1, \ldots, z_d$ forms a regular sequence in S. We may replace $x_1, \ldots, x_d$ by the images of $z_1, \ldots, z_d$ in S/Q. Since Q is a minimal prime of $J = (y_1, \ldots, y_h)$, we have that Q is nilpotent in $(S/J)_Q$. Thus, there exists $c \notin Q$ and $q_0 = p^{e_0}$ such that $cQ^{[q_0]} \subseteq J$. Now, let $r \in (x_1, \ldots, x_t) :_R x_{t+1}$, so that $x_{t+1} r = \sum_{i=1}^t r_i x_i$. Lift the relation to S, so that there are lifts $s, s_1, \ldots, s_t \in S$ of $r, r_1, \ldots, r_t$ such that $z_{t+1}s - \sum_{i=1}^t s_i z_i \in Q$. For $q \geqslant q_0$, multiplying by c and taking q-th powers we get $c(sz_{t+1})^q - \sum_{i=1}^t c(s_i z_i)^q \in cQ^{[q]} \subseteq J$. Rewriting the relation we get $cs^q z_{t+1}^q \in (z_1^q, \ldots, z_t^q, y_1, \ldots, y_h)S$. Since $z_1^q, \ldots, z_t^q, z_{t+1}^q, y_1, \ldots, y_h$ form a regular sequence in S, we have that $cs^q \in (z_1^q, \ldots, z_t^q, y_1, \ldots, y_h)S :_S z_{t+1}^q = (z_1^q, \ldots, z_t^q, y_1, \ldots, y_h)S$ for all $q \geqslant q_0$. Mapping to $S/Q = R$ this gives that $cr^q \in (x_1, \ldots, x_t)^{[q]}$ for all $q \geqslant q_0$, that is, $r \in (x_1, \ldots, x_t)^*$. $\square$

Theorem 3.7. *Let R be a weakly F-regular ring which is the homomorphic image of a Cohen-Macaulay ring. Then R is Cohen-Macaulay.*

Proof. Both issues are local (at maximal ideals), so we may assume that $(R, \mathfrak{m})$ is local. Let $x_1, \ldots, x_d$ be any system of parameters for R. By Theorem 3.6, and since R is weakly F-regular, we have that $(x_1, \ldots, x_t) :_R x_{t+1} \subseteq (x_1, \ldots, x_t)^* = (x_1, \ldots, x_t)$. Therefore R is Cohen-Macaulay. $\square$

Remark 3.8. One may wonder whether the condition of being homomorphic image of a Cohen-Macaulay local ring is very restrictive or not. In fact, it turns out that a large class of rings satisfies this assumption. First of all, observe that if $(R, \mathfrak{m}, k)$ is local and complete, then even more is true, namely by Cohen's Structure Theorem R is a quotient of a power series ring $S = k[\![x_1, \ldots, x_n]\!]$. By a result of Gabber (Theorem 4.8), in positive characteristic the same is true for a larger class of rings, called F-finite rings, that we are going to introduce in Section 4.

Recall that, if S is a ring and R is a subring of S, then R is said to be a direct summand of S if the inclusion $R \subseteq S$ splits as a map of R-modules. This means, that there exists an R-linear map $\rho : S \to R$ such that, if $i : R \hookrightarrow S$ denotes the inclusion, then $\rho \circ i = \mathrm{id}_R$. Equivalently, there exists an R-module M such that $S \cong R \oplus M$.

We now prove that direct summands of (weakly) F-regular rings are (weakly) F-regular. As a consequence of this fact and of Theorem 2.10, we have that direct summands of regular rings are F-regular.

Proposition 3.9. *Let $R \subseteq S$ be an inclusion of integral domains such that R is a direct summand of S. If S is (weakly) F-regular, then R is (weakly) F-regular.*

Proof. After possibly localizing R and S at a multiplicatively closed system $W \subseteq R$, we may only prove that R is weakly F-regular if S is. First, we recall that since R is a direct summand of S every ideal I of R is contracted from S, that is $IS \cap R = I$. In fact, $I \subseteq IS \cap R$ always holds. Conversely, if $r \in R$ is such that $r = \sum_{j=1}^{t} i_j s_j$ for some $i_j \in I$ and $s_j \in S$, then applying the splitting $\rho : S \to R$ we get $r = \rho(r) = \rho\left(\sum_{j=1}^{t} i_j s_j\right) = \sum_{j=1}^{t} i_j \rho(s_j) \in I$.

Now, let $I \subseteq R$ be an ideal. We prove that I is tightly closed. Take $x \in I^*$, this implies that there exists $c \in R^\circ$ such that $cx^q \in I^{[q]}$ for all $q = p^e \gg 0$. It follows that $cx^q \in I^{[q]}S = (IS)^{[q]}$ for all $q \gg 0$. Note that $R^\circ = R \smallsetminus \{0\} \subseteq S^\circ = S \smallsetminus \{0\}$. It follows that $x \in (IS)^* = IS$ since S is weakly F-regular by assumption. Therefore $x \in IS \cap R = I$ which shows that I is tightly closed. $\square$

Thanks to the previous result we can produce many examples of weakly F-regular rings that are not regular.

Example 3.10 (Invariant rings of finite groups are weakly F-regular)**.** Let k be an algebraically closed field of characteristic $p > 0$. We consider a power series ring $S = k[\![x_1, \ldots, x_d]\!]$ and a finite group G acting linearly on S such that $p \nmid |G|$. We denote by $R = S^G$ the corresponding invariant ring. In this case, the Reynolds operator

$$\rho : S \to R, \quad \rho(x) = \frac{1}{|G|} \sum_{\sigma \in G} \sigma(x)$$

gives a splitting for the inclusion $R \subseteq S$. So R is a direct summand of S, which is regular. Therefore, R is weakly F-regular by Proposition 3.9 and Theorem 2.10. We exhibit two explicit examples in dimension 2. Let $S = k[\![u, v]\!]$.

(1) We consider a cyclic group G of order n, generated by a matrix $\mathrm{diag}(\xi, \xi)$, where $\xi \in k$ is a primitive n-th root of unity. The group G acts linearly on each variable as $u \mapsto \xi u$ and $v \mapsto \xi v$. Then, the corresponding invariant ring R is the n-th Veronese subring

$$R = k\left[\!\left[u^n, u^{n-1}v, \ldots, uv^{n-1}, v^n\right]\!\right] = k\left[\!\left[u^i v^j \mid i + j = n\right]\!\right].$$

More generally, Veronese subrings of power series rings over an algebraically closed field are weakly F-regular. In particular, they are Cohen-Macaulay and normal.
(2) We consider again a cyclic group G of order n, but this time generated by a matrix $\operatorname{diag}(\xi, \xi^{-1})$, where $\xi \in k$ is a primitive n-th root of unity. The action of G on S is given by $u \mapsto \xi u$ and $v \mapsto \xi^{-1}v$. So, the invariant ring R is

$$R = k \llbracket u^n, uv, v^n \rrbracket \cong k\llbracket x, y, z\rrbracket/(y^n - xz)$$

which is called A_{n-1}-singularity. Note that this ring is clearly Cohen-Macaulay, since it is a hypersurface, and it is also normal by Proposition 3.5.

Remark 3.11. Every diagonal action of a group G generated by a matrix $\operatorname{diag}(\xi^i, \xi^j)$ acting linearly on $S = k\llbracket u, v\rrbracket$ produces a ring of invariants $R = S^G$ which is a direct summand of S, regardless of the characteristic of k and the order of G. In fact, consider the map of monoids $\mathbb{N}^2 \to \mathbb{Z}/(n)$ sending $(a, b) \mapsto ai + bj$, and let H be its kernel. Then $R = k\llbracket H\rrbracket$, and its complement $M = k\llbracket \mathbb{N}^2 \smallsetminus H\rrbracket$ is a module over R, giving a direct sum decomposition $S = R \oplus M$ as R-modules. This fact can, of course, be generalized in several ways; e.g., for higher number of variables, and more general group actions which can be related to maps between monoids.

4. A brief discussion on F-finiteness

Definition 4.1. We say that a ring R of characteristic $p > 0$ is *F-finite* if the Frobenius map $F : R \to R$ is a finite morphism.

Note that F is finite if and only if F^e is finite for some (equivalently, all) integers $e > 0$. In the notation introduced in Section 1, we have that R is F-finite if and only if $F^e_*(R)$ is a finitely generated R-module for some (equivalently, all) $e > 0$.

Remark 4.2. When R is reduced, R is F-finite if and only if R is a finitely generated R^q-module for some (equivalently, all) $q = p^e$, if and only if or $R^{1/q}$ is a finitely generated R-module for some (equivalently, all) $q = p^e$.

Example 4.3. A field k is F-finite if and only if $[k : k^p] < \infty$. For instance, any finitely generated field extension of a perfect field is F-finite (e.g., $k = \mathbb{F}_p(t_1, \ldots, t_n)$ is F-finite, since $[k : k^p] = p^n$). An example of a field which is not F-finite is $\mathbb{F}_p(t_1, \ldots)$.

Proposition 4.4. *The following classes of rings are F-finite:*
(1) *Polynomial rings in finitely many variables over F-finite rings.*
(2) *Power series rings in finitely many variables over F-finite rings.*
(3) *Quotients of F-finite rings.*
(4) *Localizations of F-finite rings.*

Proof. For (1): if R is F-finite and x is a variable, then $\{x^i \mid 0 \leqslant i < p\}$ is a basis of $R[x]$ as an $R[x^p]$-module. If $\{F_*(r_j) : j = 1, \ldots, n\}$ is a finite generating set of $F_*(R)$ as an R-module, then $\{F_*(r_j x^i) \mid j = 1, \ldots, n, i = 0, \ldots, p-1\}$ is then a generating of $F_*(R[x])$ as an $R[x]$-module. The proof of (2) is completely analogous, and we omit it.

For (3): assume that R is F-finite, and let $I \subseteq R$ be an ideal. By assumption we have a surjection $R^{\oplus n} \to F_*(R) \to 0$ for some $n > 0$, which induces a surjection $(R/I)^{\oplus n} \to F_*(R)/IF_*(R) \to 0$. We claim that $IF_*(R)$ is a submodule of $F_*(I)$. In fact, an element of $IF_*(R)$ is of the form $\sum_j i_j F_*(r_j)$, with $i_j \in I$ and $r_j \in R$. Recall that $i_j F_*(r_j) = F_*(i_j^p r_j) \in$

$F_*(I)$. Therefore we have a surjection $F_*(R)/IF_*(R) \to F_*(R)/F_*(I) \to 0$. Observing that $F_*(R)/F_*(I) \cong F_*(R/I)$, and combining this with the previous considerations, gives a surjection $(R/I)^{\oplus n} \to F_*(R/I) \to 0$. Therefore R/I is F-finite.

For (4): using the same argument as in (3), we have a surjection $R^{\oplus n} \to F_*(R) \to 0$. If W is a multiplicatively closed system, then localizing we obtain a surjection $(R_W)^{\oplus n} \to (F_*(R))_W \to 0$, and recalling that $F_*(R)_W \cong F_*(R_W)$ by Proposition 1.10 completes the proof. $\square$

Remark 4.5. Proposition 4.4 yields that a finitely generated k-algebra is F-finite if and only if k is F-finite, if and only if $[k : k^p] < \infty$. The same is true for complete local rings. In fact, by Cohen's structure Theorem, if $(R, \mathfrak{m})$ is a complete local ring with residue field $k = R/\mathfrak{m}$, then R is a quotient of a power series ring $S = k[\![x_1, \ldots, x_n]\!]$. By Proposition 4.4 we have that R is F-finite if and only if S is F-finite, if and only if $[k : k^p] < \infty$.

One may wonder whether, as for weakly F-regular, being F-finite is a local property, that is whether it is true that a ring R is F-finite if and only every localization $R_\mathfrak{m}$ at a maximal ideal $\mathfrak{m} \subseteq R$ is F-finite. While R F-finite implies $R_\mathfrak{m}$ F-finite by Proposition 4.4, the other direction does not always hold. An example was provided recently by Dumitrescu and Ionescu [DI20].

Example 4.6 (Dumitrescu–Ionescu)**.** Let $p > 2$ be a prime and let k be an algebraically closed field of characteristic p. Consider the ring

$$R = k\left[X, \frac{1}{\sqrt{(X+a)^3} + \sqrt{b^3}} \;\middle|\; a, b \in k,\ b \neq 0\right],$$

where X is an indeterminate, and for each square root in the denominator one chooses one of its values. Then $R_\mathfrak{m}$ is F-finite for any maximal ideal $\mathfrak{m}$ of R, but R is not F-finite.

Lemma 4.7. *Let $(R, \mathfrak{m})$ be an F-finite local ring. If R is reduced then $\widehat{R}$ is reduced.*

Proof. Since $F^e_*(R)$ is finitely generated, we have $F^e_*(R) \otimes_R \widehat{R} \cong \widehat{F^e_*R}$. Now, if we identify $F^e_*(R)$ with R, then $\widehat{F^e_*R}$ is the completion of R with respect to the ideal $F^e(\mathfrak{m})R = \mathfrak{m}^{[p^e]}$. Since $\sqrt{\mathfrak{m}^{[p^e]}} = \mathfrak{m}$, this is the same as the $\mathfrak{m}$-adic completion of R, that is $\widehat{F^e_*R} \cong F^e_*(\widehat{R})$. If R is reduced, the map $R \to F^e_*(R)$ is injective, and so $\widehat{R} \to F^e_*(R) \otimes \widehat{R} \cong F^e_*(\widehat{R})$ is injective as well, since the functor $M \mapsto M \otimes_R \widehat{R}$ is exact. Hence $\widehat{R}$ is reduced. $\square$

We conclude the section with an important theorem by Gabber that shows that F-finite rings are homomorphic image of regular rings. We record the result here without proof (see [MP21, Theorem 12.5] for a proof).

Theorem 4.8. [Gab04] *Let R be an F-finite ring. Then there exists an F-finite regular ring S such that $R = S/I$ for some ideal $I \subseteq S$.*

5. Splittings, F-pure and strongly F-regular rings

We start by giving another proof that regular rings are F-regular, to give a flavor of what is coming next. For simplicity, we only show that $R = \mathbb{F}_p[\![x_1, \ldots, x_n]\!]$ is weakly F-regular. Let $I \subseteq R$ be an ideal, and $x \in R$ be such that $cx^q \in I^{[q]}$ for some $c \neq 0$ and all $q = p^e \gg 0$. Equivalently, we have that $F^e_*(c)x \in IF^e_*(R)$ for all $e \gg 0$. Since $c \neq 0$, for all $e \gg 0$ we have that $c \notin \mathfrak{m}^{[p^e]}$ or, equivalently, $F^e_*(c) \notin \mathfrak{m}F^e_*(R)$. By Nakayama's Lemma, this simply means that $F^e_*(c)$ can be made part of a minimal generating set of the R-module $F^e_*(R)$, for $e \gg 0$. However, we have already observed that $F^e_*(R)$ is a free R-module. Therefore, for e sufficiently large we can make $F^e_*(c)$ part of a basis of $F^e_*(R)$. For a fixed $e \gg 0$, define an R-linear map $\psi : F^e_*(R) \to R$ by sending $F_*(c) \mapsto 1$, and every other element of a basis to any element of R (e.g., to 0). Then we have that

$$x = \psi(F^e_*(c))x = \psi(F^e_*(c)x) \in \psi(IF^e_*(R)) = I\psi(F^e_*(R)) \subseteq IR.$$

It follows that $I = I^*$ for every ideal $I \subseteq R$, and thus R is weakly F-regular.

In general, $F^e_*(R)$ is not a free R-module, therefore we do not have such freedom in choosing a map $\psi : F^e_*(R) \to R$. This is what motivates the study of the R-modules $\operatorname{Hom}_R(F^e_*(R), R)$, for $e \in \mathbb{N}$.

Elements of $\operatorname{Hom}_R(F^e_*(R), R)$ are called Cartier maps, or p^{-e}-linear maps, since an R-linear map $\psi : F^e_*(R) \to R$ can also be viewed as an additive map $\psi' : R \to R$ satisfying $\psi'(r^{p^e}s) = r\psi'(s)$ for all $r, s \in R$. Cartier maps can be put together to form a non-commutative $\mathbb{F}_p$-algebra $\mathcal{C} = \bigoplus_{e \geqslant 0} \operatorname{Hom}_R(F^e_*(R), R)$, called the Cartier algebra of R, whose product we briefly describe below. We will not explore this direction in these notes.

Given $\psi \in \operatorname{Hom}_R(F^e_*(R), R)$ and $\varphi \in \operatorname{Hom}_R(F^{e'}_*(R), R)$, we can multiply ψ and φ as follows: $\psi \cdot \varphi = \psi \circ F^e_*(\varphi) \in \operatorname{Hom}_R(F^{e+e'}_*(R), R)$. Note that R is not central in $\mathcal{C}$ since for $r \in R$ and $\psi \in \operatorname{Hom}_R(F^e_*(R), R)$ we have $[r \cdot \psi](F^e_*(s)) = \psi(rF^e*(s)) = \psi(F^e_*(r^{p^e}s))$, while $[\psi \cdot r](F^e_*(s)) = \psi(F^e_*(r)F^e_*(s)) = \psi(F^e_*(rs))$.

Note that $\operatorname{Hom}_R(F^e_*(R), R)$ is actually an $F^e_*(R)$-module, with module structure given by pre-multiplication: for $\psi : F^e_*(R) \to R$, and $F^e_*(r) \in F^e_*(R)$, then $[F^e_*(r) \cdot \psi] = \psi(F^e_*(r)-)$ is the map defined as $F^e_*(r) \cdot \psi(F^e_*(s)) = \psi(F^e_*(r)F^e_*(s)) = \psi(F^e_*(rs))$ for all $F^e_*(s) \in F^e_*(R)$.

In order to study Cartier maps, we recall the following general notions.

Let $f : R \to S$ be ring homomorphism. We say that f splits if there exists $g \in \operatorname{Hom}_R(S, R)$ such that $g \circ f = \operatorname{id}_R$. Note that, in particular, f has to be injective (and g surjective). On the other hand, f is said to be pure if, for every R-module M, then induced map $f_M : R \otimes_R M \to S \otimes_R M$ is injective.

Remark 5.1. It is immediate to see that if f is split, then it is pure. In fact, if $g : S \to R$ is a splitting, then if M is an R-module the map f_M still has a splitting, namely the map g_M induced by tensoring g with M. Moreover, if f is pure then $f = f_R$ must be injective.

We now apply these notions to the Frobenius homomorphism.

Definition 5.2. Let R be a ring of characteristic $p > 0$. We say that R is F-split if the Frobenius map splits. We say that R is F-pure if $F : R \to R$ is a pure homomorphism.

Remark 5.3. Using the Frobenius push-forward point of view, we have that R is F-split (resp. F-pure) if and only if the map $R \to F_*(R)$ splits (resp. is pure). By what we have observed above, F-split rings are F-pure. Moreover, F-pure rings are reduced by Proposition 1.5, since if the Frobenius map is pure it is injective.

Even though F-split and F-pure are different notions (for instance, see [DS16]), they are actually equivalent notions for large classes of rings; in particular, for complete local rings and for F-finite rings. The fact that a pure morphism $f : R \to S$ is split is true more generally whenever f is a finite map. The proof is based on the following homological lemma.

Lemma 5.4. *Let R be a ring, and M be a finitely generated R-module, with presentation $G \xrightarrow{\Theta} H \to M \to 0$ (here, G and H are finite free R-module). Let $M' = \operatorname{coker}(\Theta')$, where $(-)' = \operatorname{Hom}_R(-, R)$, and $\Theta' : H' \to G'$ is the map dual to Θ. Given a short exact sequence $0 \to A \to B \to C \to 0$ of R-modules, we have*

$$\operatorname{coker}(\operatorname{Hom}_R(M, B) \to \operatorname{Hom}_R(M, C)) = \ker(M' \otimes_R A \to M' \otimes_R B).$$

Proof. Since G and H are free, we have an exact diagram

$$\begin{tikzcd}
0 \arrow[r] & H' \otimes_R A \arrow[r] \arrow[d] & H' \otimes_R B \arrow[r] \arrow[d] & H' \otimes_R C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G' \otimes_R A \arrow[r] & G' \otimes_R B \arrow[r] & G' \otimes_R C \arrow[r] & 0
\end{tikzcd}$$

Since H' is free, we have an isomorphism $H' \otimes_R B \cong \operatorname{Hom}_R(H, B)$. Under this isomorphisms, we have $\ker(H' \otimes_R B \to G' \otimes_R B) = \ker(\operatorname{Hom}_R(H, B) \to \operatorname{Hom}_R(G, B)) = \operatorname{Hom}_R(M, B)$. Similarly for C. As $\operatorname{coker}(H' \otimes_R A \to G' \otimes_R A) \cong M' \otimes_R A$ (and similarly for B), the snake lemma concludes the proof. □

Corollary 5.5. *F-finite F-pure rings are F-split.*

Proof. If R is F-pure, then the map $\varphi : R \to F_*(R)$ is injective. Let C be the cokernel, and consider the short exact sequence $\mathcal{E} : 0 \to R \to F_*(R) \xrightarrow{\beta} C \to 0$. Apply Lemma 5.4 to $M = C$ to get that $\operatorname{coker}(\operatorname{Hom}_R(C, F_*(R)) \xrightarrow{\beta_*} \operatorname{Hom}_R(C, C)) = \ker(C' \otimes_R R \to C' \otimes_R F_*(R))$, where $\beta_* = \operatorname{Hom}_R(C, \beta)$. Note that $\ker(C' \otimes_R R \to C' \otimes_R F_*(R)) = 0$, since $R \to F_*(R)$ is pure. It follows that the map $\operatorname{Hom}_R(C, F_*(R)) \overset{\beta_*}{\operatorname{Hom}}_R(C, C)$ is surjective and, in particular, there exists $\gamma : C \to F_*(R)$ such that $\beta_*(\gamma) = \beta \circ \gamma = \operatorname{id}_C$. Then the exact sequence $\mathcal{E}$ splits, and R is therefore F-split. □

Proposition 5.6. *If R is a regular ring, then R is F-pure. Moreover, if R is F-finite, then it is F-split.*

Proof. The second claim follows immediately from the first and Corollary 5.5. For the first, thanks to Kunz's theorem 2.7 it suffices to show that a faithfully flat ring map $f : R \to S$ is pure. Let M be an R-module, and let $x \in M$ be such that $f_M(x) = 0$ in $M \otimes_R S$. Since f is flat, the inclusion $xR \subseteq M$ induces an inclusion $xR \otimes_R S \to M \otimes_R S$, under which the element $x \otimes 1$ maps to the element $f_M(x) = 0$. In particular, $x \otimes 1 = 0$, which implies that $xR \otimes_R S = 0$. By Proposition 2.4 (2) we conclude that $xR = 0$, that is, $x = 0$. Thus f_M is injective, and the proof is complete. □

We will see that even weakly F-regular rings are F-pure, but we delay the proof until Section 7.

We now show that F-purity localizes, and that F-purity can be checked locally. If R is F-finite, then the same holds for F-splitness, in light of Corollary 5.5; however, in the following result we do not assume that R is F-finite.

Proposition 5.7. *Let R be a ring. The following conditions are equivalent:*

(1) *R is F-pure.*
(2) *R_W is F-pure for every multiplicatively closed system $W \subseteq R$.*
(3) *R_P is F-pure for all prime ideals $P \in \operatorname{Spec}(R)$.*

(4) $R_{\mathfrak{m}}$ *is F-pure for all maximal ideals* $\mathfrak{m}$ *of* R.

Proof. Assume (1), and let M be an R_W-module; we want to show that $(R_W \to F_*(R_W)) \otimes_{R_W} M$ is injective. Since $F_*(R_W) \cong (F_*(R))_W$, and

$$\begin{aligned}(R_W \to F_*(R_W)) \otimes_{R_W} M &\cong ((R \to F_*(R)) \otimes_R R_W) \otimes_{R_W} M \\ &\cong (R \to F_*(R)) \otimes_R (R_W \otimes_{R_W} M) \cong (R \to F_*(R)) \otimes_R M,\end{aligned}$$

we conclude by our assumption that R is F-pure. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are trivial. Now assume (4). By way of contradiction, assume that there exists an R-module M such that $F_M : (R \to F_*(R)) \otimes_R M$ is not injective. In particular, we can find a non-zero element $x \in M$ such that $F_M(x) = 0$. By assumption, we have that $\operatorname{ann}_R(x)$ is a proper ideal, hence contained in some maximal ideal $\mathfrak{m}$. In particular, the image of x inside $M_{\mathfrak{m}} \cong R_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} M_{\mathfrak{m}}$ is still non-zero while the image of $F_M(x)$ is, of course, still zero in $(F_*(R) \otimes_R M)_{\mathfrak{m}} \cong F_*(R) \otimes_R M_{\mathfrak{m}} \cong F_*(R_{\mathfrak{m}}) \otimes_{R_{\mathfrak{m}}} M_{\mathfrak{m}}$. In other words, the map $(R_{\mathfrak{m}} \to F_*(R_{\mathfrak{m}})) \otimes_{R_{\mathfrak{m}}} M_{\mathfrak{m}}$ is not injective, contradicting the fact that $R_{\mathfrak{m}}$ is F-pure. $\square$

We observe that the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) of Proposition 5.7 hold also for F-splitness, regardless of the F-finite assumption. If R is F-finite, then an alternative proof of (4) $\Rightarrow$ (1) is also given by the following observation: R is F-split if and only if the map $\operatorname{Hom}_R(F_*(R), R) \to \operatorname{Hom}_R(R, R)$ induced by the Frobenius map $R \to F_*(R)$ is surjective. The latter happens if and only if such a map is surjective when localized at all maximal ideals $\mathfrak{m}$, and since R is F-finite, this is equivalent to $\operatorname{Hom}_R(F_*(R), R)_{\mathfrak{m}} \cong \operatorname{Hom}_{R_{\mathfrak{m}}}(F_*(R_{\mathfrak{m}}), R_{\mathfrak{m}}) \to \operatorname{Hom}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}, R_{\mathfrak{m}})$ being surjective for all maximal ideals of R, i.e., $R_{\mathfrak{m}}$ is F-split for all maximal ideals $\mathfrak{m}$ of R. While there are short exact sequences of (necessarily infinitely generated) modules which are locally split but do not split globally, we do not know whether $R_{\mathfrak{m}}$ F-split for all maximal ideals $\mathfrak{m}$ of R implies that R is F-split in general.

Lemma 5.8. *For* $e > 0$ *let* $\varphi_e : R \to F_*^e(R)$ *be the map sending* $1 \mapsto F_*^e(1)$. *The following conditions are equivalent.*

(1) *R is F-split, i.e.,* φ_1 *splits.*
(2) *There exists* $e > 0$ *such that* φ_e *splits.*
(3) φ_e *splits for all* $e > 0$.
(4) *There exists* $e > 0$ *and a surjective R-linear map* $F_*^e(R) \to R$.

Proof. Clearly (1) implies (2) and (3) implies (1). Assume (2), and let $e > 0$ be such that φ_e splits, and let $\psi_e : F_*^e(R) \to R$ be a splitting. First, we prove that $\varphi_{ne} : R \to F_*^{ne}(R)$ splits for every $n > 0$. In fact, it is enough to observe that $\varphi_{ne} = F_*^{(n-1)e}(\varphi_e) \circ \ldots \circ F_*^e(\varphi_e) \circ \varphi_e$. Thus, the composition $\psi_{ne} = \psi_e \circ F_*^e(\psi_e) \circ \ldots \circ F_*^{(n-1)e}(\psi_e)$ gives a splitting of φ_{ne}.

Now let $e' > 0$ be any integer, and choose n such that $e' < ne$. The composition

$$R \xrightarrow{\varphi_{e'}} F_*^{e'}(R) \xrightarrow{F_*^{e'}(\varphi_{ne-e'})} F_*^{e'}(F_*^{ne-e'}(R)) \cong F_*^{ne}(R) \xrightarrow{\psi_{ne}} R$$

is the identity, and therefore $\psi_{e'} = \psi_{ne} \circ F_*^{e'}(\varphi_{ne-e'})$ is a splitting of $\varphi_{e'}$.

Finally, clearly (2) implies (4). Conversely, if $\psi : F_*^e(R) \to R$ is a surjective map, then let $F_*^e(r) \in F_*(R)$ be an element that maps to 1. Note that the composition

$$R \xrightarrow{\varphi_e} F_*^e(R) \xrightarrow{\cdot F_*^e(r)} F_*^e(R) \xrightarrow{\psi} R,$$

where $F^e_*(R) \xrightarrow{\cdot F^e_*(r)} F^e_*(R)$ is the $F^e_*(R)$-linear map consisting of multiplication by the element $F^e_*(r)$, is the identity. Therefore R is F-split. $\square$

From now on, assume that R is F-finite. By Theorem 4.8, we can write $R = S/I$ for some F-finite regular ring S. Using this presentation of R, we want to describe more explicitly elements of $\operatorname{Hom}_R(F^e_*(R), R)$. We will focus on the case $e = 1$, but the theory we develop and the results we present apply with the obvious modifications to any power of p.

Remark 5.9. If S is a Gorenstein local ring, then clearly $\omega_S \cong S$, where ω_S denotes the canonical module of S. Since $F_*(S)$ is an S-module of the same dimension as S, we have that $\omega_{F_*(S)} \cong \operatorname{Hom}_S(F_*(S), S)$; however, as ring $F_*(S)$ is isomorphic to S, and hence it is itself Gorenstein. It follows that $\operatorname{Hom}_S(F_*(S), S) \cong F_*(S)$ as a $F_*(S)$-module.

The next Lemma allows to identify a generator of $\operatorname{Hom}_S(F_*(S), S)$ as a $F_*(S)$-module in the regular (local and $\mathbb{N}$-graded) case.

Lemma 5.10. *Let k be a perfect field, and S be either a power series ring $k[\![x_1, \ldots, x_n]\!]$ or a positively graded polynomial ring $k[x_1, \ldots, x_n]$. Let $\Lambda = \{(i_1, \ldots, i_n) \in \mathbb{N}^n \mid 0 \leqslant i_j < p\}$, and $\mathcal{B} = \{F_*(x_1^{i_1} \cdots x_n^{i_n}) \mid \underline{i} = (i_1, \ldots, i_n) \in \Lambda\}$ be the standard basis of $F_*(S)$ as an S-module. For $\underline{i} \in \Lambda$ let $\varphi_{\underline{i}} : F_*(S) \to S$ be the S-linear map defined on the elements of $\mathcal{B}$ as follows:*

$$\varphi_{\underline{i}}(F_*(x_1^{j_1} \cdots x_n^{j_n})) = \begin{cases} 1 & \text{if } \underline{j} = \underline{i} \\ \\ 0 & \text{otherwise} \end{cases}$$

Let $\Phi = \varphi_{(p-1,\ldots,p-1)}$. Then $\operatorname{Hom}_S(F_*(S), S)$ is a principal $F_*(S)$-module, generated by Φ.

Proof. Since $F_*(S)$ is free with basis $\mathcal{B}$, it is clear that $\operatorname{Hom}_S(F_*(S), S)$ is a free S-module, with dual basis given by $\{\varphi_{\underline{i}} \mid \underline{i} \in \mathcal{B}\}$. Thus, it suffices to show that $\varphi_{\underline{i}} \in [F_*(S) \cdot \Phi]$ for every $\underline{i} \in \Lambda$. Recall that the $F_*(S)$-action is defined as follows: if $F_*(s) \in F_*(S)$ and $\varphi \in \operatorname{Hom}_S(F_*(S), S)$, then $[F_*(s) \cdot \varphi] = \varphi(F_*(s) \cdot -)$ is the map such that $[F_*(s) \cdot \varphi](F_*(s')) = \varphi(F_*(s)F_*(s')) = \varphi(F_*(ss'))$. Thus, it is sufficient to observe that

$$\varphi_{\underline{i}}(-) = \Phi(F_*(x_1^{p-1-i_1} \cdots x_n^{p-1-i_n}) \cdot -) = [F_*(x_1^{p-1-i_1} \cdots x_n^{p-1-i_n}) \cdot \Phi] \in F_*(S) \cdot \Phi. \qquad \square$$

The map Φ is called the trace.

Proposition 5.11. *Let $(S, \mathfrak{m})$ be either an F-finite complete regular local ring with perfect residue field k, or an F-finite graded polynomial ring over a perfect field k. Let $I \subseteq S$ be an ideal, homogeneous in the second case, and let $R = S/I$. There is an isomorphism of $F_*(R)$-modules:*

$$\Theta : \frac{IF_*(S) :_{F_*(S)} F_*(I)}{IF_*(S)} \longrightarrow \operatorname{Hom}_R(F_*(R), R)$$

$$F_*(s) \longmapsto [F_*(s) \cdot \Phi] = \Phi(F_*(s)-)$$

Proof. We only give a proof in the case when $(S, \mathfrak{m})$ is complete. The proof in the graded case is completely analogous. If we let $S/\mathfrak{m} = k$, then we may assume that $S = k[\![x_1, \ldots, x_n]\!]$. It is easy to see that Θ is $F_*(S)$-linear; moreover, since it maps $F_*(I)$ to zero, it is also $F_*(R) \cong F_*(S)/F_*(I)$-linear.

Now observe that if $s \in (IF_*(S) :_{F_*(S)} F_*(I))$, then the S-linear map $[F_*(s) \cdot \Phi] \in \operatorname{Hom}_S(F_*(S), S)$ induces an $R = S/I$-linear map $F_*(R) \to R$. To see this, it suffices to verify that $[F_*(s) \cdot \Phi]$ maps $F_*(I)$ into I. But this comes from our choice of s, since $F_*(s)F_*(I) \subseteq IF_*(S)$, and therefore

$$[F_*(s) \cdot \Phi](F_*(I)) = \Phi(F_*(s)F_*(I)) \subseteq \Phi(IF_*(S)) = I\Phi(F_*(S)) \subseteq I,$$

where we used that Φ is S-linear. On the other hand, if $F_*(s) \in IF_*(S)$, then $\Phi(F_*(s)F_*(S)) \subseteq I\Phi(F_*(S)) \subseteq I$, that is, $\Phi(F_*(s)-)$ induces the zero map. This shows that Θ is well-defined.

We now prove that Θ is surjective. Let $\psi \in \operatorname{Hom}_R(F_*(R), R)$. Since $F_*(R) \cong F_*(S)/F_*(I)$, then ψ can be identified with an S-linear map $F_*(S) \to S/I$ that maps $F_*(I)$ to zero. Consider the short exact sequence $0 \to I \to S \to S/I \to 0$. Apply the functor $\operatorname{Hom}_S(F_*(S), -)$ to get an exact sequence

$$0 \longrightarrow \operatorname{Hom}_S(F_*(S), I) \to \operatorname{Hom}_S(F_*(S), S) \longrightarrow \operatorname{Hom}_S(F_*(S), S/I) \longrightarrow \operatorname{Ext}^1_S(F_*(S), I).$$

Since S is regular, $F_*(S)$ is a finitely generated flat S-module by Kunz's Theorem 2.7, and hence it is free. In particular, $\operatorname{Ext}^1_S(F_*(S), I) = 0$, and thus we conclude that ψ comes from an S-linear map $\varphi : F_*(S) \to S$ which maps $F_*(I)$ to I. By Lemma 5.10 we have that $\varphi = F_*(s) \cdot \Phi$ for some $s \in S$. We claim that $F_*(s)F_*(I) \subseteq IF_*(S)$. If not, assume that $F_*(r) \in F_*(I)$ is such that $F_*(s)F_*(r) = F_*(sr) \notin IF_*(S)$. Using the notation of Lemma 5.10 we have that $\mathcal{B} = \{F_*(x_1^{i_1} \cdots x_n^{i_n}) \mid \underline{i} \in \Lambda\}$ is an S-basis of $F_*(S)$, and therefore we can write $F_*(sr) = \sum_{\underline{i} \in \Lambda} s_{\underline{i}} F_*(x_1^{i_1} \cdots x_n^{i_n})$ for some $s_{\underline{i}} \in S$. By hypothesis, there is $\underline{j} \in \Lambda$ such that $s_{\underline{j}} \notin I$. Now consider $r' = r x_1^{p-1-j_1} \cdots x_n^{p-1-j_n}$, so that $F_*(r')$ is still an element of $F_*(I)$. Note that $\Phi(F_*(sr')) = \Phi(s_{\underline{j}} x_1^{p-1} \cdots x_n^{p-1}) = s_{\underline{j}} \notin I$. This contradicts the fact that $[F_*(s) \cdot \Phi]$ maps $F_*(I)$ to I.

Finally, to show that Θ is injective, assume that $\Phi(F_*(s)-)$ is the zero map, and by way of contradiction suppose that $F_*(s) \notin IF_*(S)$. Repeating the same argument as above with $r = 1$ we see that $F_*(s) = \sum_{\underline{i} \in \Lambda} s_{\underline{i}} F_*(x_1^{i_1} \cdots x_n^{i_n})$, and there is $\underline{j} \in \Lambda$ such that $s_{\underline{j}} \notin I$. If we let $r' = x_1^{p-1-j_1} \cdots x_n^{p-1-j_n}$, then $\Phi(F_*(sr')) = \Phi(s_{\underline{j}} x_1^{p-1} \cdots x_n^{p-1}) = s_{\underline{j}} \notin I$, which contradicts the fact that $[F_*(s) \cdot \Phi]$ is the zero map. $\square$

Remark 5.12. For practical purposes, it is more convenient to identify $IF_*(S)$ and $F_*(I)$ inside $F_*(S)$ with $I^{[p]}$ and I inside S. In this way, an R-linear map $\psi : F_*(R) \to R$ corresponds to a choice of an element $s \in (I^{[p]} :_S I)/I^{[p]}$, with $\psi = \Theta(F_*(s)) = [F_*(s) \cdot \Phi]$.

Remark 5.13. If k is an F-finite field, $S = k[\![x_1, \ldots, x_n]\!]$, and $I = (f) \subseteq \mathfrak{m} = (x_1, \ldots, x_n)$, then $(I^{[p]} :_S I) = (f^{p-1})$. Thus, if we let $R = S/(f)$, any R-linear map $\psi : F_*(R) \to R$ corresponds to the choice of an element $f^{p-1}g \in S$: more specifically, $\psi = \Phi(F_*(f^{p-1}g)-)$. Moreover, such a map is zero if and only if $g \in (f)$.

Example 5.14. Let $S = \mathbb{F}_2[\![x, y, z]\!]$, $f = xz + y^2$ and $R = S/(f)$. Consider the element fy, and the corresponding Cartier map $\psi = \Phi(F_*(fy)-)$. Note that, for instance, $\psi(F_*(1)) = \Phi(F_*(xyz+y^3)) = \Phi(F_*(xyz)) + \Phi(yF_*(y)) = 1 + y\Phi(F_*(y)) = 1$, and $\psi(F_*(xz)) = \Phi(F_*(x^2yz^2)) + \Phi(F_*(xy^3z)) = \Phi(xzF_*(y)) + \Phi(yF_*(xyz)) = xz\Phi(F_*(y)) + y\Phi(F_*(xyz)) = y$.

The previous example is F-pure, since the map $\Phi(F_*(fy)-)$ is a splitting of $R \to F_*(R)$. Recall that F-pure rings are reduced; however, there are integral domains which are not F-pure.

Example 5.15. Let $S = \mathbb{F}_3[\![x,y]\!]$, $f = x^2 - y^3$ and $R = S/(f)$. Then $f^{p-1} = f^2 = x^4 + x^2y^3 + y^6$. Note that $f^2 \in \mathfrak{m}^{[3]} = (x^3, y^3)$. In particular, for all $g \in S$ we have that $\Phi(F_*(f^2g)F_*(S)) \subseteq \Phi(\mathfrak{m}F_*(S)) \subseteq \mathfrak{m}\Phi(F_*(S)) \subseteq \mathfrak{m}$. It follows that there is no surjective Cartier map $F_*(R) \to R$, and therefore R cannot be F-pure, even if it is a domain.

Example 5.16. Let $S = \mathbb{F}_p[\![x,y,z]\!]$, let $I = (xy, xz, yz)$, and $R = S/I$. Note that $(xyz)^{p-1} \in (I^{[p]} :_S I)$, and therefore $\psi(-) = \Phi(F_*((xyz)^{p-1}))$ is a Cartier map on $R = S/I$. Observe that $\psi(F_*(1)) = \Phi(F_*((xyz)^{p-1})) = 1$, and thus R is F-pure.

Theorem 5.17 (Fedder's criterion)**.** *Let $(S, \mathfrak{m})$ be either an F-finite complete regular local ring with perfect residue field k, or an F-finite graded polynomial ring over a perfect field k. Let $I \subseteq S$ be an ideal, homogeneous in the second case. The ring $R = S/I$ is F-pure if and only if $(I^{[p]} :_S I) \not\subseteq \mathfrak{m}^{[p]}$.*

Proof. We only show the case in which $S = k[\![x_1, \ldots, x_n]\!]$, as the proof in the graded case is analogous. We claim that it suffices to show that, for a given $s \in S$, one has that $s \notin \mathfrak{m}^{[p]}$ if and only if $\Theta(F_*(s)) = [F_*(s) \cdot \Phi]$ is a surjective map. In fact, suppose we have proven this claim. If $R = S/I$ is F-pure, then there exists a splitting $\psi : F_*(R) \to R$ on the Frobenius map, which is necessarily surjective. By 5.11 and Remark 5.12, we have that $\psi = \Theta(F_*(s))$ for some $s \in (I^{[p]} :_S I)$, and by the claim $s \notin \mathfrak{m}^{[p]}$. It follows that $(I^{[p]} :_S I) \not\subseteq \mathfrak{m}^{[p]}$. Conversely, if there exists $s \in (I^{[p]} :_S I) \smallsetminus \mathfrak{m}^{[p]}$, then the R-linear map $\psi = \Theta(F_*(s)) : F_*(R) \to R$ is surjective by the claim. It follows by Lemma 5.8 that R is F-pure.

We therefore prove the claim. Observe that, since S is local, $\Theta(F_*(s))$ is surjective if and only if its image is not contained in the maximal ideal $\mathfrak{m}$. If $s \in \mathfrak{m}^{[p]}$, then $F_*(s) \in \mathfrak{m}F_*(S)$, and therefore $\Phi(F_*(s)F_*(S)) \subseteq \Phi(\mathfrak{m}F_*(S)) \subseteq \mathfrak{m}\Phi(S) \subseteq \mathfrak{m}$. Conversely, if $s \notin \mathfrak{m}^{[p]}$, we can find an element $r \in S$ such that $rs = (x_1 \cdots x_n)^{p-1} + g$, with $g \in \mathfrak{m}^{[p]}$. It follows that $[F_*(s) \cdot \Phi](F_*(r)) = \Phi(F_*((x_1 \cdots x_n)^{p-1}) + F_*(g)) = 1 + \Phi(F_*(g)) \in 1 + \Phi(\mathfrak{m}F_*(S)) \subseteq 1 + \mathfrak{m}$, as desired. □

Example 5.18. Let $R = \mathbb{F}_p[x_1, \ldots, x_n]$ or $R = \mathbb{F}_p[\![x_1, \ldots, x_n]\!]$, and I be a monomial ideal. We show that R/I is F-pure (equivalently, F-split) if and only if I is squarefree. The "only if" direction is clear, since F-pure rings are reduced, and the quotient by a monomial ideal is reduced if and only if the ideal is squarefree. Conversely, observe that the monomial $(x_1 \cdots x_n)^{p-1}$ always belongs to $(I^{[p]} :_S I)$, because every minimal generator of I divides $x_1 \cdots x_n$, and $(x_1 \cdots x_n)^{p-1} \notin \mathfrak{m}^{[p]}$. We then conclude by Fedder's criterion, Theorem 5.17.

Example 5.19. Let $S = \mathbb{F}_7[\![x,y,z]\!]$, let $f = x^3 + y^3 + z^3$ and $R = S/(f)$. If $p = 7$, then the element $f^{p-1} = f^6$ contains the monomial $x^6y^6z^6$ in its support, with non-zero coefficient. Thus, we can find $\lambda \in \mathbb{F}_p$ such that $\lambda f^6 = x^6y^6z^6 + g$, with $g \in \mathfrak{m}^{[7]} = (x^7, y^7, z^7)$. Consider the Cartier map $\psi = \Phi(F_*(\lambda f^6)-)$; then $\psi(F_*(1)) = \Phi(F_*(x^6y^6z^6)) + \Phi(F_*(g)) = 1 + \Phi(F_*(g)) \in 1 + \Phi(\mathfrak{m}F_*(S)) \subseteq 1 + \mathfrak{m}$. It follows that ψ is surjective (in this case, one can actually check that $\psi(F_*(1)) = 1$), and thus R is F-pure.

If instead of $\mathbb{F}_7$ we choose $\mathbb{F}_5$ as the base field, then the same argument does not work, since $f^{p-1} = f^4 \in \mathfrak{m}^{[5]}$. Fedder's criterion implies that R is not F-pure in this case.

In general, for $f = x^3 + y^3 + z^3 \in S_p = \mathbb{F}_p[\![x,y,z]\!]$ with $p > 3$, one can show that $f^{p-1} \notin \mathfrak{m}^{[p]}$ if and only if $p \equiv 1 \bmod 3$.

Example 5.20. Let $S = \mathbb{F}_p[\![x,y,z]\!]$, let $f = xz - y^n$ for $n \geqslant 2$, and $R = S/(f)$. We have already proved that R is weakly F-regular (in fact, F-regular), since it is a direct summand of a regular ring. To see that it is F-split, observe that f^{p-1} contains in its support the monomial $(xz)^{p-1} \notin \mathfrak{m}^{[p]}$. It follows from Fedder's criterion, Theorem 5.17, that R is F-split. If we consider the map $\psi(-) = \Phi(F_*(f^{p-1}y^{p-1})-) \in \operatorname{Hom}_R(F_*(R), R)$, then one can check that $\psi(F_*(1)) = 1$, so that ψ is a splitting of $1 \mapsto F_*(1)$. Observe that, more generally, we can easily find maps $\psi_j(-) = \Phi(F_*(f^{p-1}y^{p-1-j})-)$ that split $1 \mapsto F_*(y^j)$ for all $0 \leqslant j \leqslant p-1$.

Fedder's criterion holds more generally for rings that are quotient of a regular local ring (not necessarily complete with perfect residue field) or a graded quotient of a graded polynomial ring over a field (not necessarily perfect). Moreover, the assumption that R is a quotient of a regular ring is not very restrictive. This is clear when R is complete by Cohen's structure theorem or when R is F-finite by Gabber's Theorem 4.8.

We now turn our attention to strongly F-regular rings. We recall that we are always assuming that R is F-finite, even when we do not write it explicitly. We start with an example

Example 5.21. Let $S = \mathbb{F}_p[\![x,y,z]\!]$, let $f = xz - y^n$ for $n \geqslant 2$, and $R = S/(f)$, as in Example 5.20. We have already shown that we can find splittings of $1 \mapsto F_*(y^j)$ for all $0 \leqslant j \leqslant p-1$. Clearly there is no hope to split the map $1 \mapsto F_*(y^p) = yF_*(1)$, since any R-linear map $\psi : F_*(R) \to R$ will be such that $\psi(yF_*(1)) = y\psi(F_*(1)) \in (y)$. However, we can split $1 \mapsto F^2_*(y^p)$ (in fact, we will see that, in this ring, for every $c \neq 0$ there exists $e \gg 0$ such that $1 \mapsto F^e_*(c)$ splits). Along these lines, we now show that, for $e \gg 0$, one can split $1 \mapsto F^e_*(x)$ and, with a symmetric strategy, $1 \mapsto F^e_*(z)$. Let $e \gg 0$ be such that $p^e > n$. Note that $(xz)^{p^e-2}y^n$ appears with coefficient $p^e - 1 = -1 \neq 0$ in the expansion of f^{p^e-1}, and therefore $x^{p^e-2}(yz)^{p^e-1}$ appears with the same coefficient in the expansion of $g = -f^{p-1}zy^{p^e-1-n}$. The map $\psi(-) = \Phi^e(F^e_*(g)-) \in \operatorname{Hom}_R(F^e_*(R), R)$ is then such that $\psi(F^e_*(x)) = \Phi^e(F^e_*(gx)) = \Phi^e(F^e_*((xyz)^{p^e-1})) = 1$. We will show that, in this ring R, for $e \gg 0$ one can actually split $1 \mapsto F^e_*(c)$ for any $c \neq 0$.

Definition 5.22. An F-finite ring R is said to be *strongly F-regular* if, for every $c \in R^\circ$, there exists $e > 0$ such that the map $R \to F^e_*(R)$ sending $1 \mapsto F^e_*(c)$ splits.

Some easy facts, that follow from the definitions and what we have proved so far are:

- Strongly F-regular are F-pure (choosing $c = 1$ in the definition). In particular, they are reduced.
- If there exists $c \in R^\circ$ and $e > 0$ such that the map $1 \mapsto F^e_*(c)$ splits, then R is F-pure. In fact, the splitting is a surjective Cartier map, and R is F-pure by Lemma 5.8.
- If the map $1 \mapsto F^e_*(c)$ splits for some $e > 0$, then $1 \mapsto F^{e'}_*(c)$ splits for every $e' \geqslant e$. In fact, assume that $1 \mapsto F^e_*(c)$ has a splitting ψ_e. Since R is F-pure, for every $e'' > 0$ we have a splitting $\gamma_{e''}$ of the map $1 \mapsto F^{e''}_*(1)$. For every $e' > e$, we have that

$$\begin{array}{ccccc} F^{e'}_*(R) & \xrightarrow{F^{e'-e}_*(\psi_e)} & F^{e'-e}_*(R) & \xrightarrow{\gamma_{e'-e}} & R \\ F^{e'}_*(c) & \longrightarrow & F^{e'-e}_*(1) & \longrightarrow & 1 \end{array}$$

is a splitting of $1 \mapsto F^{e'}_*(c)$.

The main reason why strong F-regular singularities were introduced by Hochster and Huneke is to that, contrary to weak F-regularity, strong F-regularity localizes, as we will show next. We first record a useful remark.

Remark 5.23. Let R be a ring, and M, N be finitely generated R-modules. Given a multiplicatively closed system W and a map $\psi \in \mathrm{Hom}_{R_W}(M_W, N_W) \cong (\mathrm{Hom}_R(M,N))_W$, we can find $\varphi \in \mathrm{Hom}_R(M,N)$ and $w \in W$ such that $\varphi = w\psi$. In particular, if R is an F-finite ring and W a multiplicatively closed system, then given a map $\psi : F_*(R_W) \to R_W$ we can find a map $\varphi : F_*(R) \to R$ and an element $w \in W$ such that $\varphi = w\psi$.

Proposition 5.24. *Let R be an F-finite ring. The following are equivalent:*

(1) *R is strongly F-regular*
(2) *R_W is strongly F-regular for every multiplicatively closed system W.*
(3) *R_P is strongly F-regular for all $P \in \mathrm{Spec}(R)$.*
(4) *$R_{\mathfrak{m}}$ is strongly F-regular for every maximal ideal $\mathfrak{m}$ of R.*

Proof. Let $\mathrm{Min}(R) = \{P_1, \ldots, P_s\}$. First assume that R is strongly F-regular, and let W be a multiplicatively closed system. Let $\frac{c}{w} \in R_W$ be an element not in any minimal prime of R_W. Assume that $P_1, \ldots, P_t$ are the minimal primes of R which contain c, and $P_{t+1}, \ldots, P_s$ are those which do not. Observe that $P_1, \ldots, P_t$ cannot be minimal primes of R_W, that is, $P_j \cap W \neq \emptyset$ for any $j = 1, \ldots, t$. If we pick $c' \in (P_{t+1} \cap \ldots \cap P_s) \smallsetminus (P_1 \cup \ldots \cup P_t)$, then the image of c' in R_W is zero (since R is F-pure, hence reduced) and, in particular, $\frac{c}{w} = \frac{c+c'}{w}$ in R_W. Moreover, because of our choice we have that $c+c' \in R^\circ$. Since R is strongly F-regular, there exists $e > 0$ such that the map $R \to F_*^e(R)$ sending $1 \mapsto F_*^e(c+c')$ has a splitting, say ψ, which induces a map $\psi_W : F_*^e(R_W) \to R_W$ sending $F_*^e\left(\frac{c+c'}{1}\right) \mapsto \frac{1}{1}$. The compositions

$$\begin{array}{ccccc} F_*^e(R_W) & \xrightarrow{\cdot F_*^e(w)} & F_*^e(R_W) & \xrightarrow{\psi_W} & R_W \\ F_*^e\left(\frac{c}{w}\right) = F_*^e\left(\frac{c+c'}{w}\right) & \longrightarrow & F_*^e\left(\frac{c+c'}{1}\right) & \longrightarrow & \frac{1}{1} \end{array}$$

give the desired splitting of the map $R_W \to F_*^e(R_W)$ sending $\frac{1}{1} \mapsto F_*^e\left(\frac{c}{w}\right)$.

The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are trivial. Now assume that $R_{\mathfrak{m}}$ is strongly F-regular for every maximal ideal $\mathfrak{m}$. Let $c \in R^\circ$, and for every maximal ideal $\mathfrak{m}$ choose $e(\mathfrak{m}) > 0$ such that the map $R_{\mathfrak{m}} \to F_*^{e(\mathfrak{m})}(R_{\mathfrak{m}})$ sending $\frac{1}{1} \mapsto F_*^{e(\mathfrak{m})}\left(\frac{c}{1}\right)$ splits. Let $\psi_{\mathfrak{m}}$ be a splitting. By Remark 5.23 there exists $f_{\mathfrak{m}} \notin \mathfrak{m}$ and a map $\varphi_{\mathfrak{m}} \in \mathrm{Hom}_R(F_*^{e(\mathfrak{m})}(R), R)$ such that $f_{\mathfrak{m}}\psi_{\mathfrak{m}} = \varphi_{\mathfrak{m}}$. As a consequence, $\varphi_{\mathfrak{m}}(F_*^{e(\mathfrak{m})}(c)) = f_{\mathfrak{m}}$. Consider the ideal $J = (f_{\mathfrak{m}} \mid \mathfrak{m}$ maximal ideal of $R)$. Then $J = R$, since no maximal ideal of R contains it. Therefore, there exist maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ and elements $r_1, \ldots, r_n \in R$ such that $\sum_{i=1}^n r_i f_{\mathfrak{m}_i} = 1$. Let $e = \max\{e(\mathfrak{m}_1), \ldots, e(\mathfrak{m}_n)\}$, and $e_i = e - e(\mathfrak{m}_i)$. Since $R_{\mathfrak{m}}$ is F-pure for all maximal ideals $\mathfrak{m}$, it is F-pure by Proposition 5.7. We can then find splittings $\gamma_{e_i} : F_*^{e_i}(R) \to R$ of the map $1 \mapsto F_*^{e_i}(1)$. Then we get compositions

$$\begin{array}{ccccccc} F_*^e(R) & \xrightarrow{F_*^{e_i}(\varphi_{\mathfrak{m}_i})} & F_*^{e_i}(R) & \xrightarrow{\cdot F_*^{e_i}(f_{\mathfrak{m}_i}^{p^{e_i}-1})} & F_*^{e_i}(R) & \xrightarrow{\gamma_{e_i}} & R \\ F_*^e(c) & \longrightarrow & F_*^{e_i}(f_{\mathfrak{m}_i}) & \longrightarrow & F_*^{e_i}(f_{\mathfrak{m}_i}^{p^{e_i}}) = f_{\mathfrak{m}_i}F_*^{e_i}(1) & \longrightarrow & f_{\mathfrak{m}_i} \end{array}$$

which we call δ_i. We finally claim that $\delta = \sum_{i=1}^n r_i\delta_i$ is the desired splitting. In fact, we have that $\delta(F_*^e(c)) = \sum_{i=1}^n r_i f_{\mathfrak{m}_i} = 1$, as desired. ☐

We will now show that F-finite regular rings are strongly F-regular, as a consequence of Kunz's theorem, and that strongly F-regular rings are F-regular.

Theorem 5.25. *Let R be an F-finite ring. If R is regular, then R is strongly F-regular. If R is strongly F-regular, then it is F-regular; in particular, strongly F-regular rings are Cohen-Macaulay and normal.*

Proof. By Proposition 5.24 we may assume that $(R, \mathfrak{m})$ is local. By Kunz's Theorem, $F_*^e(R)$ is a flat R-module for every integer $e > 0$, finitely generated by assumption. Therefore $F_*^e(R)$ is projective, and hence free, since R is local. Let $c \in R^\circ$, and let $e \gg 0$ such that $c \notin \mathfrak{m}^{[p^e]}$. Then $F_*^e(c) \notin \mathfrak{m}F_*^e(R)$, that is, $F_*^e(c)$ is a minimal generator of $F_*^e(R)$, by Nakayama's lemma. Since $F_*^e(R)$ is free, $F_*^e(c)$ can actually be made into part of a basis. Thus, we may define a n R-linear map $F_*^e(R) \to R$ which sends $F_*^e(c)$ to 1 and every other basis element to any element of R; for instance, to zero. This is the desired splitting of $1 \mapsto F_*^e(c)$.

For the second claim, thanks to Proposition 5.24, it suffices to show that every ideal of a strongly F-regular ring is tightly closed. To prove this, we repeat the argument that we sketched at the beginning of the section: if $I \subseteq R$ is an ideal, and $x \in I^*$, then there exists $c \in R^\circ$ such that $F_*^e(c)x \in IF_*^e(R)$ for all $e \gg 0$. But for $e \gg 0$ we can find $\psi \in \operatorname{Hom}_R(F_*^e(R), R)$ such that $\psi(F_*^e(c)) = 1$. It follows that $x = \psi(F_*^e(c)x) \in \psi(IF_*^e(R)) \subseteq I$.

Since every F-finite ring is the homomorphic images of a regular ring by 4.8, strongly F-regular rings are Cohen-Macaulay and normal by Proposition 3.5 and Theorem 3.7. $\square$

The next result is an extension of Proposition 3.9.

Proposition 5.26. *Let $R \subseteq S$ be F-finite domains such that R is a direct summand of S. If S is strongly F-regular, then R is strongly F-regular.*

Proof. We fix a $c \in R^\circ \subseteq S^\circ = S \setminus \{0\}$. Since S is strongly F-regular, there exists $e > 0$ and an S-linear map $\psi_e : F_*^e(S) \to S$ which is a splitting for the map $S \to F_*^e(S)$ sending $1 \mapsto F_*^e(c)$, that is $\psi_e(F_*^e(c)) = 1$. Let $\rho : S \to R$ be a splitting for the inclusion $R \subseteq S$. Then the composition $\rho \circ \psi_e : F_*^e(S) \to R$ is an R-linear map sending $F_*^e(c) \mapsto 1$. Restricting this map to $F_*^e(R)$ yields an R-linear splitting $F_*^e(R) \to R$ such that $F_*^e(c) \mapsto 1$. Therefore, R is strongly F-regular. $\square$

The previous result allows us to construct many examples of strongly F-regular rings.

Example 5.27. Let k be an F-finite field of characteristic $p > 0$.

(1) We consider a power series ring $S = k[\![x_1, \ldots, x_d]\!]$ and a finite group G acting linearly on S such that $p \nmid |G|$. As we saw in Example 3.10, R is a direct summand of S, which is regular (so strongly F-regular by Theorem 5.25), hence R is strongly F-regular as well. In particular, all rings of Example 3.10 are also strongly F-regular.

(2) Let $R = k[x, y, u, v]/(xy - uv)$. Then $R \cong k[a, b]\#k[c, d] \cong k[ac, bd, ad, bc]$ is a direct summand of $S = k[a, b, c, d]$. Therefore, R is strongly F-regular. More generally, Segre products of polynomial rings are strongly F-regular.

Another way to see that Example 5.27 (2) is strongly F-regular is given by the following very useful result.

Proposition 5.28. *Let R be an F-finite ring, and $c \in R^\circ$. If R_c is strongly F-regular and there exists $e_0 \geqslant 1$ such that the map $R \to F_*^{e_0}(R)$ sending $1 \mapsto F_*^{e_0}(c)$ splits, then R is strongly F-regular.*

Proof. Observe that R is F-split by Lemma 5.8 (4). Let $d \in R^\circ$; since R_c is strongly F-regular, and $d \in (R_c)^\circ$, there exists $e' > 0$ and an R_c-linear map $F_*^{e'}(R_c) \to R_c$ sending $\frac{F_*^{e'}(d)}{1} \mapsto \frac{1}{1}$. By Remark 5.23, there exists $N \geqslant 0$ and a map $\gamma : F_*^{e'}(R) \to R$ sending $F_*^{e'}(d) \mapsto c^N$.

Let φ_e denote the splitting of $1 \mapsto F_*^e(1)$. For $e \geqslant 0$, since $F_*^{e'}(\varphi_e)$ is $F_*^{e'}(R)$-linear, it sends $F^{e+e'}(d^{p^e}) = F_*^{e'}(d) \cdot F_*^{e+e'}(1)$ to $F_*^{e'}(d)$. Define $\gamma_{e+e'}$ as the following composition

$$\begin{array}{ccccccc} F_*^{e+e'}(R) & \xrightarrow{\cdot F_*^{e+e'}(d^{p^e-1})} & F_*^{e+e'}(R) & \xrightarrow{F_*^{e'}(\varphi_e)} & F_*^{e'}(R) & \longrightarrow & R \\ F_*^{e+e'}(d) & \longmapsto & F_*^{e+e'}(d^{p^e}) & \longmapsto & F_*^{e'}(d) & \longmapsto & c^N \end{array}$$

Relabeling, this gives a set of maps $\gamma_e \in \operatorname{Hom}_R(F_*^e(R), R)$ such that $\gamma_e(F_*^e(d)) = c^N$ for all $e \geqslant e'$.

In a similar way, since $F_*^{e_0}(\varphi_e)$ is $F_*^{e_0}(R)$-linear, it sends $F^{e+e_0}(c^{p^e}) = F_*^{e_0}(c) \cdot F_*^{e+e_0}(1)$ to $F_*^{e_0}(c)$. In particular, for all $p^e > N$, if we define by δ_{e+e_0} the composition

$$\begin{array}{ccccccc} F_*^{e+e_0}(R) & \xrightarrow{\cdot F_*^{e+e_0}(c^{p^e-N})} & F_*^{e+e_0}(R) & \xrightarrow{F_*^{e_0}(\varphi_e)} & F_*^{e_0}(R) & \longrightarrow & R \\ F_*^{e+e_0}(c^N) & \longmapsto & F^{e+e_0}(c^{p^e}) & \longmapsto & F_*^{e_0}(c) & \longmapsto & 1 \end{array}$$

then relabeling this gives a set of maps $\delta_e \in \operatorname{Hom}_R(F_*^e(R), R)$ such that $\delta_e(F_*^e(c^N)) = 1$ for all $e \geqslant e_0$ such that $p^e > N$. For $e \gg 0$, we can therefore consider the map $\psi = \delta_e \circ F_*^e(\gamma_e) \in \operatorname{Hom}_R(F_*^{2e}(R), R)$, which is such that $\psi(F_*^{2e}(d)) = \delta_e(F_*^e(c^N)) = 1$. $\square$

Example 5.29. Let $S = \mathbb{F}_3[X]$, where $X = \begin{bmatrix} x & v \\ u & y \end{bmatrix}$, and let $R = S/(\det(X))$, as in Example 5.27 (2). By Fedder's criterion, an R-linear map $\varphi : F_*(R) \to R$ corresponds to an element in $(\det(X)^{[3]} :_S \det(X)) = (\det(X)^2) = (x^2y^2 + xyuv + u^2v^2)$. Consider the element $\alpha = yuv\det(X)^2$, and the map $\varphi(-) = \Phi(F_*(\alpha)-)$. One can readily check that $\varphi(F_*(x)) = 1$, and therefore φ splits the map $1 \mapsto F_*(x)$. Since

$$R_x \cong \frac{\mathbb{F}_3[x, y, u, v, x^{-1}]}{(y - uvx^{-1})} \cong \mathbb{F}_3[x, u, v]_x$$

is regular, hence strongly F-regular, the ring R is strongly F-regular by Proposition 5.28.

6. F-injectivity and F-rationality

6.1. **A short recap on local cohomology.** We briefly recall a few ways to define local cohomology modules supported at an ideal. Let R be a ring, and $I \subseteq R$ be an ideal, generated by $x_1, \ldots, x_t$. Let M be an R-module. The i-th local cohomology module of M with support in I, denoted, $H_I^i(M)$, can be equivalently defined in one of the following ways:

(1) The Čech complex

$$\check{\mathrm{C}}^\bullet : 0 \longrightarrow M \xrightarrow{\partial^0} \bigoplus_{i=1}^t M_{x_i} \xrightarrow{\partial^1} \bigoplus_{i<j} M_{x_ix_j} \longrightarrow \ldots \qquad \longrightarrow M_{x_1\cdots x_t} \longrightarrow 0$$

is the complex whose maps are (up to sign) just the natural localization maps. For instance, if $t=2$, then

$$\partial^0(m) = \left(\frac{m}{1}, \frac{m}{1}\right) \in M_{x_1} \oplus M_{x_2},$$

and

$$\partial^1\left(\frac{m_1}{x_1^a}, \frac{m_2}{x_2^b}\right) = \frac{m_1}{x_1^a} - \frac{m_2}{x_2^b} = \frac{m_1x_2^b - m_2x_1^2}{x_1^ax_2^b} \in M_{x_1x_2}.$$

One can check that, with appropriate choices of sign, the one above is indeed a complex. Then $H^i_I(M) = H^i(\check{\mathrm{C}}^\bullet)$.

(2) For every N let $K^\bullet_N = K^\bullet(x_1^N, \ldots, x_t^N)$ be the Koszul complex on $x_1^N, \ldots, x_t^N$. There the direct limit $\lim\limits_{n\to\infty}\left(K_1^\bullet \to K_2^\bullet \to \ldots K_n^\bullet \to K_{n+1}^\bullet \to \ldots\right) = K_\infty^\bullet$ coincides with the Čech complex $\check{\mathrm{C}}^\bullet$, so that $H^i_I(M) = H^i(K^\bullet_\infty \otimes_R M)$. For example, for $t=2$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] \\
\ldots \arrow[r] & M \arrow[r, equal] \arrow[d, "\begin{bmatrix} x_1^n \\ x_2^n \end{bmatrix}"'] & M \arrow[r] \arrow[d, "\begin{bmatrix} x_1^{n+1} \\ x_2^{n+1} \end{bmatrix}"] & \ldots & M \arrow[d] \\
\ldots \arrow[r] & M^{\oplus 2} \arrow[r, "\begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}"] \arrow[d, "\begin{bmatrix} x_2^n & -x_1^n \end{bmatrix}"'] & M^{\oplus 2} \arrow[r] \arrow[d, "\begin{bmatrix} x_2^{n+1} & -x_1^{n+1} \end{bmatrix}"] & \ldots & M_{x_1} \oplus M_{x_2} \arrow[d] \\
\ldots \arrow[r] & M \arrow[r, "\cdot x_1x_2"] \arrow[d] & M \arrow[r] \arrow[d] & \ldots & M_{x_1x_2} \arrow[d] \\
 & 0 & 0 & & 0
\end{tikzcd}$$

(3) The surjections $\ldots \to R/I^{n+1} \to R/I^n \to \ldots \to R/I^2 \to R/I$ induce maps $\mathrm{Ext}^i_R(R/I, M) \to \mathrm{Ext}^i_R(R/I^2, M) \to \ldots \mathrm{Ext}^i_R(R/I^n, M) \to \mathrm{Ext}^i_R(R/I^{n+1}, M) \to \ldots$ (not necessarily injections). Then $H^i_I(M) = \lim\limits_{n\to\infty} \mathrm{Ext}^i_R(R/I^n, M)$.

(4) Let $\Gamma_I(-)$ be the left-exact covariant I-torsion functor, defined as $\Gamma_I(M) = \bigcup_N (0 :_M I^N)$. Then $H^i_I(M)$ is the i-th right derived functor of $\Gamma_I(-)$.

Remark 6.1. Let $I = (x_1, \ldots, x_t)$ be an ideal of R and let M be an R-module.

- If follows immediately from the last two definitions of $H^i_I(M)$ that this module only depends on the radical of the ideal I, i.e., $H^i_I(M) \cong H^i_{\sqrt{I}}(M)$.
- It follows from the first definition that $H^i_I(M) = 0$ for all $i \geqslant t+1$ and

$$H^t_I(M) = M_{x_1\cdots x_t} / \sum_{i=1}^t \mathrm{Im}\left(M_{x_1\cdots x_{i-1}x_{i+1}\cdots x_t}\right).$$

6.2. **F-injectivity.** If $\varphi : R \to S$ is a ring homomorphism, and $I = (x_1, \ldots, x_t) \subseteq R$ is an ideal, then there is a natural map $\check{\mathrm{C}}^\bullet(x_1, \ldots, x_t; R) \to \check{\mathrm{C}}^\bullet(\varphi(x_1), \ldots, \varphi(x_t); S)$, which gives a map on local cohomology modules: $H^i_I(\varphi) : H^i_I(R) \to H^i_{IS}(S)$, where we notice that $IS = (\varphi(x_1), \ldots, \varphi(x_t))S$. In our case of interest, we will consider a local ring $(R, \mathfrak{m})$, the Frobenius map $F : R \to R$, and $I = \mathfrak{m}$. Note that $F(\mathfrak{m})R = \mathfrak{m}^{[p]}$ has the same radical as $\mathfrak{m}$. Therefore $H^i_{\mathfrak{m}^{[p]}}(R) \cong H^i_{\mathfrak{m}}(R)$. This fact, together with the previous discussion, gives Frobenius maps $H^i_{\mathfrak{m}}(F) : H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R)$ on local cohomology modules. If no confusion arises, we will denote each map $H^i_{\mathfrak{m}}(F)$ still by F.

Even if R is reduced, that is, F is injective, there is no guarantee that F is injective on $H^i_{\mathfrak{m}}(R)$.

Example 6.2. Consider the one-dimensional local domain $R = \mathbb{F}_5[\![x, y]\!]/(x^2 - y^3)$. Since R is Cohen-Macaulay, the only non-vanishing local cohomology module is $H^1_{\mathfrak{m}}(R)$. Moreover, since $\sqrt{(x)} = \mathfrak{m}$, we can compute it using the Čech complex on x, thus we obtain $H^1_{\mathfrak{m}}(R) \cong R_x/R$. Note that $\eta = \left[\frac{y^2}{x}\right] \neq 0$ in $H^1_{\mathfrak{m}}(R)$: otherwise, there would exist $r \in R$ such that $\frac{y^2}{x} = \frac{r}{1}$ and, since R is a domain, it would immediately follow that $y^2 \in (x)$ inside R. But this is clearly false. However, note that $F(\eta) = \left[\frac{y^{10}}{x^5}\right] = 0$, since $y^{10} = y(y^3)^3 = y(x^2)^3 \in (x^5)$ in R.

Definition 6.3. A local ring $(R, \mathfrak{m})$ is said to be *F-injective* if the Frobenius map $F : H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R)$ is injective for every i. A ring R is F-injective if $R_{\mathfrak{m}}$ is F-injective for every maximal ideal $\mathfrak{m}$ of R.

Since $H^i_{\mathfrak{m}}(R) \cong H^i_{\widehat{\mathfrak{m}}}(\widehat{R})$, and the Frobenius map is the same whether we consider the local cohomology as a module over R or over $\widehat{R}$, we immediately have that R is F-injective if and only if $\widehat{R}$ is F-injective.

Remark 6.4. If $(R, \mathfrak{m})$ is an F-injective ring of positive dimension, then $H^0_{\mathfrak{m}}(R) = 0$. Otherwise, there would exists $0 \neq r \in \mathfrak{m}$ such that $\mathfrak{m}^N r = 0$ for $N \gg 0$. However, $F : H^0_{\mathfrak{m}}(R) \to H^0_{\mathfrak{m}}(R)$ is injective, and thus $F^e(r) \neq 0$ for all $e > 0$. But then $F^e(r) = r^{p^e} \in \mathfrak{m}^{p^e-1} r = 0$ for $e \gg 0$, a contradiction.

Proposition 6.5. *Let R be an F-pure ring, then R is F-injective.*

Proof. Both notions are local at maximal ideals, so we can assume without loss of generality that $(R, \mathfrak{m})$ is local. We prove the statement only in the case when R is F-finite. By Corollary 5.5, R is F-split, so there exists a splitting $\rho : F_*(R) \to R$ such that $\rho \circ F = \mathrm{id}_R$, where $F : R \to F_*(R)$ is the Frobenius homomorphism. The map ρ induces a natural map on local cohomology modules $H^i_{\mathfrak{m}}(\rho) : H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R)$ for any i. By functoriality, this map is such that $H^i_{\mathfrak{m}}(\rho) \circ H^i_{\mathfrak{m}}(F) = \mathrm{id}_{H^i_{\mathfrak{m}}(R)}$. So the map $H^i_{\mathfrak{m}}(F) = F : H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R)$ splits, hence it is injective. □

We are going to prove that F-injectivity localizes. To do so, we need local duality, which we briefly recall here. For more details, see for instance [BS13].

Let $(S, \mathfrak{n})$ be a complete Gorenstein local ring of dimension d (of any characteristic). We denote by $E_S(S/\mathfrak{n})$ the injective hull of the residue field $S/\mathfrak{n}$, that is the smallest injective module containing $S/\mathfrak{n}$. Equivalently, $E_S(S/\mathfrak{n})$ is an injective S-module and an essential extension of $S/\mathfrak{n}$, i.e., for any submodule H of $E_S(S/\mathfrak{n})$, if $H \cap S/\mathfrak{n} = 0$ then $H = 0$. The injective hull of $S/\mathfrak{n}$ exists and is unique up to isomorphism. The functor $(-)^\vee =$

$\operatorname{Hom}_S(-, E_S(S/\mathfrak{n}))$ on the category of S-modules is called Matlis dual functor. It gives isomorphisms

$$H^i_{\mathfrak{n}}(M)^\vee \cong \operatorname{Ext}^{d-i}_S(M, S),$$

for any $0 \leqslant i \leqslant d$ and any finitely generated S-module M. When $(R, \mathfrak{m}, k)$ is a complete Cohen-Macaulay local ring of dimension d with canonical module ω_R, and M is a finitely generated R-module, Matlis duality gives rise to the following isomorphisms known as Grothendieck's local duality (see [BH93, Theorem 3.5.8]):

$$H^i_{\mathfrak{m}}(M) \cong \operatorname{Hom}_R(\operatorname{Ext}^{d-i}_R(M, \omega_R), E(k)), \text{ and}$$
$$\operatorname{Ext}^i_R(M, \omega_R) \cong \operatorname{Hom}_R(H^{d-i}_{\mathfrak{m}}(M), E(k)).$$

The last isomorphism with $M = R$ and $i = 0$ yields the fact that the Matlis dual of the top-dimensional local cohomology module of the ring is isomorphic the canonical module:

$$H^d_{\mathfrak{m}}(R)^\vee \cong \operatorname{Hom}_R(R, \omega_R) \cong \omega_R.$$

Proposition 6.6. *Let $(R, \mathfrak{m})$ be an F-finite F-injective local ring. Then R_W is F-injective for every multiplicatively closed system W. In particular, R is reduced.*

Proof. By definition, F-injectivity is tested locally at maximal ideals. So to prove that R_W is F-injective it is enough to prove that $(R_W)_{\mathfrak{m}}$ is F-injective for any maximal ideal $\mathfrak{m}$ of R_W. But any prime ideal of R_W is just a prime ideal P of R that does not intersect W, so $(R_W)_{\mathfrak{m}} \cong R_P$. Therefore we may assume without loss of generality that $W = R \smallsetminus P$ for some $P \in \operatorname{Spec}(R)$, and prove that R_P is F-injective. We show that we can also assume that R is complete. Clearly, R F-injective implies $\widehat{R}$ F-injective. Let $P \in \operatorname{Spec}(R)$ and choose a minimal prime Q of $P\widehat{R}$ such that $\dim R_P = \dim \widehat{R}_Q$. The local ring map $R_P \to \widehat{R}_Q$ is faithfully flat. Moreover, notice that $P\widehat{R}_Q$ and $Q\widehat{R}_Q$ have the same radical. Therefore we obtain an isomorphism in local cohomology $H^i_{Q\widehat{R}_Q}(\widehat{R}_Q) \cong H^i_{PR_P}(R_P) \otimes_{R_P} \widehat{R}_Q$ for all i which is compatible with the Frobenius action. Thus, if the Frobenius is injective on $H^i_{Q\widehat{R}_Q}(\widehat{R}_Q)$, i.e., if $\widehat{R}_Q$ is F-injective, then it is also injective on $H^i_{PR_P}(R_P)$, i.e., R_P is F-injective.

As explained above, now we assume that $(R, \mathfrak{m})$ is a complete local ring, $P \in \operatorname{Spec}(R)$ and we prove that R_P is F-injective. By Cohen's Structure Theorem, there exists an n-dimensional regular local ring $(S, \mathfrak{n})$ such that $R = S/I$ for some ideal $I \subseteq S$. Let Q be the lift of P to S. By local duality, we have isomorphisms $H^i_{\mathfrak{m}}(R)^\vee \cong \operatorname{Ext}^{n-i}_S(R, S)$, where $(-)^\vee = \operatorname{Hom}_S(-, E_S(S/\mathfrak{n}))$ is the Matlis dual functor. Let us view the Frobenius map $F : H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R)$ as an R-linear map $\varphi : H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(F_*(R))$. Then, this map is injective if and only if its Matlis dual

$$\varphi^\vee : \operatorname{Ext}^{n-i}_S(F_*(R), S) \to \operatorname{Ext}^{n-i}_S(R, S)$$

is surjective. Therefore, if R is F-injective then $\varphi^\vee : \operatorname{Ext}^{n-i}_S(F_*(R), S) \to \operatorname{Ext}^{n-i}_S(R, S)$ is surjective for all i, and therefore $F^\vee : (\operatorname{Ext}^{n-i}_S(F_*(R), S))_Q \to (\operatorname{Ext}^{n-i}_S(R, S))_Q$ is surjective for every i. Since localization is flat, we have that $(\operatorname{Ext}^{n-i}_S(R, S))_Q \cong \operatorname{Ext}^{n-i}_{S_Q}(R_Q, S_Q)$; moreover, since R is F-finite, we have that $(\operatorname{Ext}^{n-i}_S(F_*(R), S))_Q \cong \operatorname{Ext}^{n-i}_{S_Q}(F_*(R)_Q, S_Q) \cong \operatorname{Ext}^{n-i}_{S_Q}(F_*(R_Q), S_Q)$. Note that clearly $R_Q \cong R_P$. To conclude the proof of the first part, observe that S_Q is again a regular local ring; applying local duality over S_Q we conclude that

$$\varphi : H^{\dim S_Q - n+i}_{PR_P}(R_P) \to H^{\dim S_Q-n+i}_{PR_P}(F_*(R_P))$$

is injective for all i, that is, (R_P, PR_P) is F-injective. Observe that, in order for the isomorphism $\operatorname{Ext}^{n-i}_{S_Q}(R_Q, S_Q)^\vee \cong H^{\dim S_Q-n+i}_{PR_P}(R_P)$ to hold, it is not required that S_Q and R_P are complete.

Finally, to see that R is reduced, observe first of all that F-injective rings have no embedded primes. If P was an embedded prime, then R_P would be an F-injective ring of positive dimension such that $H^0_{PR_P}(R_P) \neq 0$, which would contradict Remark 6.4. Moreover, if $P \in \operatorname{Min}(R)$, then since R_P is F-injective we have that Frobenius is injective on $H^0_{PR_P}(R_P) = R_P$. Therefore R_P is reduced for every minimal prime of R, that is, R_P is a field. Now, if we consider an irredundant primary decomposition $(0) = Q_1 \cap \ldots \cap Q_t$, then we have that each Q_i corresponds to a minimal prime P_i and, since R_{P_i} is a field, we have that $Q_iR_{P_i} = P_iR_{P_i}$. Since Q_i is P_i-primary, it follows at once that $Q_i = P_i$ for all i, and R is reduced. □

6.3. **F-rationality.** Now, we introduce the last notion of F-singularity we are going to study.

Definition 6.7. A local ring $(R, \mathfrak{m})$ of dimension d is said to be *F-rational* if it is Cohen-Macaulay, and for every $c \in R^\circ$ there exists an integer $e > 0$ such that the map $cF^e : H^d_\mathfrak{m}(R) \to H^d_\mathfrak{m}(R)$ is injective. A ring R is F-rational if $R_\mathfrak{m}$ is F-rational for every maximal ideal $\mathfrak{m}$ of R.

It is clear that F-rational rings are F-injective. Using this observation, it also becomes clear that a Cohen-Macaulay ring R is F-rational if and only if for every $c \in R^\circ$ the map $cF^e : H^d_\mathfrak{m}(R) \to H^d_\mathfrak{m}(R)$ is injective for all $e \gg 0$. In fact, if it is injective for one single $e > 0$, then for $e' > e$ the map $cF^{e'}$ is obtained as the composition of the injective maps $cF^e \circ F^{e'-e}$, where $F^{e'-e}$ is injective since R is F-injective.

6.3.1. *F-rationality and tight closure.* The original definition of F-rationality requires that every ideal generated by a system of parameters (even partial) is tightly closed. We now show that this is equivalent to the one given above for rings that are the homomorphic image of a Cohen-Macaulay ring.

We first discuss some facts about the top local cohomology modules.

Using the $K^\bullet_\infty$ definition of local cohomology, and using the right exactness of direct limits, one can see that if $I = (x_1, \ldots, x_t)$, and $x = x_1 \cdots x_t$, then $H^t_I(M)$ is the direct limit of the system

$$M/(x_1, \ldots, x_t)M \xrightarrow{\cdot x} M/(x_1^2, \ldots, x_t^2)M \xrightarrow{\cdot x} M/(x_1^3, \ldots, x_t^3)M \longrightarrow \ldots$$

When $x_1, \ldots, x_d$ is a full system of parameters, i.e., a system of parameters such that $I = (x_1, \ldots, x_d)$ is an $\mathfrak{m}$-primary ideal, then thanks to the above an element $\eta \in H^d_\mathfrak{m}(R)$ can be seen as a class $\left[\frac{r}{x^t}\right]$ for some $t \geqslant 0$ and $r \in R$, where $x = x_1 \cdots x_d$. In this way, $\eta = 0$ if and only if $\frac{r}{x^t}$ maps to zero in the direct limit as above, if and only if $x^n r \in (x_1^{n+t}, \ldots, x_d^{n+t})$ for some (equivalently, all) $n \gg 0$, if and only if $r \in (x_1^{n+t}, \ldots, x_d^{n+t}) :_R x^n$ for some (equivalently, all) $n \gg 0$. Note that, when R is Cohen-Macaulay, this is equivalent to $r \in (x_1^t, \ldots, x_d^t)$, since $x_1, \ldots, x_d$ forms a regular sequence.

Finally, when R has characteristic p, one can readily check that the Frobenius map $F : H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R)$ is such that $F(\eta) = F\left(\left[\frac{r}{x^t}\right]\right) = \left[\frac{r^p}{x^{tp}}\right]$. Therefore, if R is Cohen-Macaulay, $cF^e(\eta) = 0$ is equivalent to $cr^{p^e} \in (x_1^{tp^e}, \ldots, x_d^{tp^e})$.

For the reader's convenience, we collect the previous observations in the following lemma.

Lemma 6.8. *Let $(R, \mathfrak{m})$ be a local ring of dimension d, let $x_1, \ldots, x_d$ be a full system of parameters, let $x = x_1 \cdots x_d$, and let $\eta = \left[\frac{r}{x^t}\right] \in H^d_{\mathfrak{m}}(R)$ for $r \in R$ and $t \geqslant 0$. Then the following facts holds.*

(1) $F(\eta) = \left[\frac{r^p}{x^{tp}}\right]$.
(2) *$\eta = 0$ if and only if $r \in (x_1^{n+t}, \ldots, x_d^{n+t}) :_R x^n$ for some (equivalently, all) $n \gg 0$.*
(3) *If R is Cohen-Macaulay, then $\eta = 0$ if and only if $r \in (x_1^t, \ldots, x_d^t)$.*

Theorem 6.9. *Let $(R, \mathfrak{m})$ be a local ring of dimension d which is the homomorphic image of a Cohen-Macaulay ring. Then R is F-rational if and only if every ideal $(x_1, \ldots, x_t)$ generated by a system of parameters is tightly closed.*

Proof. First assume $d = 0$. If R is F-rational, then R is a field, since $H^0_{\mathfrak{m}}(R) = R$ and $F : R \to R$ is injective. Thus the only proper ideal is (0), and it is tightly closed. Conversely, if the only ideal generated by a system of parameters (i.e., (0)) is tightly closed, then R is reduced, and thus it is a field. So R is F-rational.

Now assume $d > 0$. Assume that R is F-rational, let $I = (x_1, \ldots, x_d)$ be any ideal generated by a full system of parameters, and let $x = x_1 \cdots x_d$. If $r \in R$ is such that $cr^q \in I^{[q]}$ for some $c \in R^\circ$ and all $q = p^e \gg 0$, then the element $\eta = \left[\frac{r}{x}\right]$ is such that $cF^e(\eta) = 0$ in $H^d_{\mathfrak{m}}(R)$ for all $e \gg 0$. Since R is F-rational, we conclude that $\eta = 0$, that is, $r \in I$, and thus $I = I^*$. Now let $x_1, \ldots, x_t$ be any system of parameters, and complete it to a full system of parameters: $x_1, \ldots, x_t, x_{t+1}, \ldots, x_d$. For all $N \geqslant 1$ we have that

$$(x_1, \ldots, x_t)^* \subseteq (x_1, \ldots, x_t, x_{t+1}^N, \ldots, x_d^N)^* = (x_1, \ldots, x_t, x_{t+1}^N, \ldots, x_d^N),$$

and therefore $(x_1, \ldots, x_t)^* \subseteq \bigcap_{N \geqslant 1} (x_1, \ldots, x_t) + (x_{t+1}^N, \ldots, x_t^N) = (x_1, \ldots, x_t)$.

Conversely, assume that every ideal generated by a system of parameters is tightly closed, and observe that R is Cohen-Macaulay by colon capturing, Theorem 3.6. First, we show that R is F-injective. If $\eta = \left[\frac{r}{x^t}\right] \in H^d_{\mathfrak{m}}(R)$ is such that $F(\eta) = \left[\frac{r^p}{x^{tp}}\right] = 0$, then we have that $r^p \in (x_1^t, \ldots, x_d^t)^{[p]}$, and this implies that $r^q \in (x_1^t, \ldots, x_d^t)^{[q]}$ for all $q = p^e$. In particular, $r \in (x_1^t, \ldots, x_d^t)^* = (x_1^t, \ldots, x_d^t)$, and therefore $\eta = 0$.

Now, let $c \in R^\circ$, and for $e > 0$ define $N_e = \ker(cF^e)$. We claim that each N_e is an R-module, and $N_{e+1} \subseteq N_e$ for all $e > 0$. Clearly each N_e is an Abelian group. If $r \in R$ and $\eta \in N_e$, then $cF^e(r\eta) = r^{p^e} cF^e(\eta) = 0$, and thus $r\eta \in N_e$. If $\eta \in N_{e+1}$, then $F(cF^e(\eta)) = c^p F^{e+1}(\eta) = 0$. Since R is F-injective, it follows that $cF^e(\eta) = 0$, and thus $\eta \in N_e$. We have a descending chain of R-submodules of $H^d_{\mathfrak{m}}(R)$:

$$N_1 \supseteq N_2 \supseteq N_3 \ldots \supseteq N_e \supseteq N_{e+1} \supseteq \ldots$$

which must eventually stabilize because $H^d_{\mathfrak{m}}(R)$ is Artinian. Let e_0 be such that $N_e = N_{e_0}$ for all $e \geqslant e_0$. If $N_{e_0} \neq 0$, that is, there exists $0 \neq \eta \in H^d_{\mathfrak{m}}(R)$ such that $cF^{e_0}(\eta) = 0$, then by what we have shown above we have that $cF^e(\eta) = 0$ for all $e \geqslant e_0$. If we write $\eta = \left[\frac{r}{x^t}\right]$, then this means that $cr^q \in I^{[q]}$ for all $q \gg 0$, where $I = (x_1^t, \ldots, x_d^t)$. Since I is tightly closed by assumption, we have that $r \in I$, so that $\eta = 0$. This shows that $N_{e_0} = 0$, so that cF^{e_0} is injective . As $c \in R^\circ$ was arbitrary, it follows that R is F-rational. □

Corollary 6.10. *Let $(R, \mathfrak{m})$ be a local ring which is the homomorphic image of a Cohen-Macaulay ring. If R is weakly F-regular, then it is F-rational. Moreover, if R is F-rational, then it is a normal domain.*

Proof. The first claim follows immediately from Theorem 6.9. For the second, observe that the proof of Proposition 3.5 that weakly F-regular rings are normal only requires that (0) and ideals generated by a single regular element are tightly closed, which is still true if R is F-rational. $\square$

Examples 6.11. It follows from Corollary 6.10 that all weakly (and strongly) F-regular rings of Examples 3.10 and 5.27 are also F-rational. An example of non F-rational singularity is given by the following ring from Example 1.19. Let $R = \mathbb{F}_p[x, y, z]/(x^2 - y^5 - z^7)$, then the ideal generated by parameters (y, z) is not tightly closed, since $x \in (y, z)^* \setminus (y, z)$. Therefore R is not F-rational.

One may wonder whether, similarly to weakly and strongly F-regular, a direct summand of an F-rational or F-injective ring is still F-rational or F-injective. This is false in general, an example has been constructed by Watanabe. We record the example here, but we refer the reader to [MP21, Remark 9.4] or to the original [Wat97] for details and proofs.

Example 6.12. Let $R = \mathbb{F}_3[x, y, z]/(x^2+y^3+z^5)$ with grading $\deg(x) = 15$, $\deg(y) = 10$, and $\deg(z) = 6$. Then R is a two-dimensional normal domain which is a direct summand of an F-rational ring. However, R is not F-injective. In fact, the cohomology class $\left[\frac{x}{yz}\right] \in H^2_{\mathfrak{m}}(R)$ is nonzero, but

$$F\left(\left[\frac{x}{yz}\right]\right) = \left[\frac{x^3}{y^3z^3}\right] = 0,$$

since $x^3 \in (y^3, z^3)R$. On the other hand, if we consider $T = \overline{\mathbb{F}}_p[x, y, z]/(x^2 + y^3 + z^5)$ where $p > 5$ is a prime number, it is well known that T is isomorphic to the invariant ring $T \cong \overline{\mathbb{F}}_p[u, v]^{\mathcal{I}}$ known as E_8-singularity. Here, $\mathcal{I} \subseteq \mathrm{GL}(2, \overline{\mathbb{F}}_p)$ is the binary icosahedral group of order 120. In particular, T is strongly F-regular and so also F-rational.

We saw that a local ring $(R, \mathfrak{m})$ is F-injective if and only if $\widehat{R}$ is F-injective. One may wonder whether the same is true for F-rational as well. One direction is easy. Namely, assume that $\widehat{R}$ is F-rational and I is an ideal of R generated by a system of parameters. Since $R \to \widehat{R}$ is a faithfully flat ring extension, by Proposition 2.8 we have

$$I^* = (I^*\widehat{R}) \cap R \subseteq (I\widehat{R})^* \cap R = (I\widehat{R}) \cap R = I.$$

Therefore I is tightly closed, and R is F-rational. The implication R F-rational $\Rightarrow$ $\widehat{R}$ F-rational holds if R is F-finite (or more generally if R is excellent). We record here the result without proof (for a proof see [BH93, Corollary 10.3.19] or [MP21, Theorem 6.16]).

Theorem 6.13. *Let $(R, \mathfrak{m})$ be a local F-finite ring. Then R is F-rational if and only if $\widehat{R}$ is F-rational.*

6.3.2. *F-rationality localizes.* Now, we show that F-rationality localizes. While this can be proved using the local cohomology definition using a strategy similar to that of Proposition 6.6, we will show it by proving the more general fact that tight closure of ideals generated by regular sequences commutes with localization.

We start with a lemma.

Lemma 6.14. *Let R be a Noetherian ring, $I \subseteq R$ be an ideal, and W be a multiplicatively closed system. There exists $s \in W$ such that, for all $m \geqslant 1$, we have*

$$\bigcup_{w \in W} (I^m :_R w) = (I^m :_R s^m).$$

Proof. Let $G = \operatorname{gr}_I(R) = \bigoplus_{n \geqslant 0} I^n/I^{n+1}$ be the associated graded ring of R with respect to I, which is a Noetherian ring. Consider the set $S = \{\operatorname{ann}_G(w) \mid w \in W\}$. Since G is Noetherian, S has a maximal element $\operatorname{ann}_G(s)$, for some $s \in W$. In particular, $\operatorname{ann}_G(s) = \operatorname{ann}_G(sw)$ for all $w \in W$, by maximality. Let $m \geqslant 1$, and $x \in \bigcup_{w \in W}(I^m :_R w)$, so that $wx \in I^m$ for some $w \in W$. Let $t \geqslant 0$ be such that $x \in I^t \smallsetminus I^{t+1}$. If $t \geqslant m$, then clearly $x \in I^m \subseteq (I^m :_R s^m)$. Otherwise, $swx = 0$ in G implies that $x \in \operatorname{ann}_G(sw) = \operatorname{ann}_G(s)$, so that $sx = 0$ in G. This implies that $sx \in I^{t+1}$. Repeating the argument with sx in place of x, eventually yields that $s^m x \in I^m$, as desired. $\square$

Corollary 6.15. *Let R be a Noetherian ring of characteristic p, and I be an ideal generated by a regular sequence of $x_1, \ldots, x_d$. Let s be as Lemma 6.14. Then*

$$\bigcup_{w \in W} (I^{[q]} :_R w) = (I^{[q]} :_R s^{(d+1)q}).$$

Proof. Let $x \in I^{[q]} :_R w$ for some $w \in W$. We prove by induction on $n \geqslant 0$ that $y_n = s^{q+n}x \in I^{[q]} + I^{q+n}$. The case $n = 0$ follows from Lemma 6.14, since $s^q x \in I^q$. Now assume that $y_n \in I^{[q]} + I^{q+n}$. Since $wy_n = s^{q+n}wx \in I^{[q]}$, we can write $wy_n = \sum_{i=1}^d r_i x_i^q$. By induction we also have $wy_n = \sum_{i=1}^d ws_i x_i^q + \sum_\alpha wr_\alpha x^\alpha$, where $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ is such that $\sum_{i=1}^d \alpha_i \geqslant q + n$, and $0 \leqslant \alpha_i < q$ for all i. We may assume that among the monomials x^α appearing in the above writing there are no repetitions. Putting the above relations together we obtain that $\sum_\alpha wr_\alpha x^\alpha = \sum_{i=1}^d (r_i - ws_i)x_i^q \in I^{[q]}$. Since the elements $x_1, \ldots, x_d$ form a regular sequence, and each monomial $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ does not belong to $I^{[q]}$, we obtain that each $wr_\alpha \in I$. It follows that $sr_\alpha \in I$ for all α, and thus $y_{n+1} = sy_n = \sum_{i=1}^d ss_i x_i^q + \sum_\alpha sr_\alpha x^\alpha \in I^{[q]} + II^{q+n} = I^{[q]} + I^{q+n+1}$, as desired. Finally, since I is generated by d elements, it is easy to see that $I^{d(q-1)+1} \subseteq I^{[q]}$, by the pigeon hole principle. It follows that $y_{dq} = s^{(d+1)q}x \in I^{[q]} + I^{(d+1)q} \subseteq I^{[q]}$, which concludes the proof that $x \in (I^{[q]} :_R s^{(d+1)q})$. $\square$

Theorem 6.16. *Let R be a ring of characteristic $p > 0$, and $x_1, \ldots, x_d$ be a regular sequence. Let $I = (x_1, \ldots, x_d)$, and W be a multiplicatively closed system. Then $I^* R_W = (IR_W)^*$. In particular, if R is an F-rational ring, then R_W is F-rational for every multiplicatively closed system W.*

Proof. We prove the first claim. Without loss of generality we may assume that $IR_W \neq R_W$, otherwise the equality is trivial. We have already observed that $I^* R_W \subseteq (IR_W)^*$ always holds. For the converse, let $\frac{x}{w} \in R_W$ be such that $\frac{c}{w'} \frac{x^q}{w^q} \in I^{[q]} R_W$ for some $\frac{c}{w'} \in (R_W)^\circ$ and all $q = p^e \gg 0$. Let $s \in W$ be as in Lemma 6.14, so that $\bigcup_{w \in W}(I^{[q]} :_R w) = (I^{[q]} :_R s^{(d+1)q})$ for all $q = p^e$. By prime avoidance, we may find $c' \in R^\circ$ such that $\frac{c'}{1} = \frac{c}{1}$ in R_W. By clearing denominators, we can find $w(q) \in W$ such that $w(q)c'x^q \in I^{[q]}$ for all $q \gg 0$. By Corollary 6.15, we have that $s^{(d+1)q}c'x^q = c'(s^{d+1}x)^q \in I^{[q]}$ for all $q \gg 0$, from which we get that $s^{d+1}x \in I^*$. It follows that $\frac{x}{w} \in I^* R_W$, as desired.

The proof of the second claim now follows from the first. In fact, if R is F-rational, then it is Cohen-Macaulay, and so is every localization at a multiplicatively closed system W. By definition, F-rationality is tested locally at maximal ideals; therefore, by the same argument of the proof of Proposition 6.6 without loss of generality we may assume that $W = R \smallsetminus P$ for some $P \in \operatorname{Spec}(R)$. If $x_1, \ldots, x_d \in R$ are such that their images in R_P form a system of parameters, then they are also a system of parameters in R, and hence a regular sequence. We have shown above that $((x_1, \ldots, x_d)R_P)^* = (x_1, \ldots, x_d)^* R_P = (x_1, \ldots, x_d)R_P$, where the last equality follows from Theorem 6.9 and our assumption that R is F-rational. This proves that ideals generated by arbitrary system of parameters in R_P are tightly closed, and therefore R_P is F-rational, again by Theorem 6.9. $\square$

6.3.3. *Smith's characterization of F-rationality.* Now, we give a third characterization of F-rationality in terms of local cohomology due to Karen Smith. This is crucial for the geometric interpretation of F-rationality, and we will also use it to prove the so called deformation property.

Definition 6.17. Let $(R, \mathfrak{m})$ be a local ring and let $N \subseteq H^i_{\mathfrak{m}}(R)$ be an R-submodule. We say that N is *F-stable* if $F(N) \subseteq N$, where $F : H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R)$ is the Frobenius action on local cohomology.

Theorem 6.18 (Smith). *Let $(R, \mathfrak{m})$ be an F-finite local ring of dimension d. Then the following are equivalent:*

(1) *R is F-rational.*
(2) *R is Cohen-Macaulay and $H^d_{\mathfrak{m}}(R)$ has no proper nonzero F-stable submodule.*

Proof. We recall that local cohomology commutes with completion, and moreover the Frobenius structure on $H^d_{\mathfrak{m}}(R)$ is unaffected when passing to the completion. So by this observation and by Theorem 6.13, we can assume without loss of generality that $(R, \mathfrak{m})$ is complete.

First, assume that (1) holds, i.e., R is F-rational. Let $N \subsetneq H^d_{\mathfrak{m}}(R)$ be an F-stable submodule. By local duality, we obtain an epimorphism $H^d_{\mathfrak{m}}(R)^{\vee} \cong \omega_R \to N^{\vee} \to 0$. Since R is a normal domain (Corollary 6.10), ω_R is torsionfree of rank 1. Therefore $N^{\vee}$, and thus N, is a torsion module. Hence, there exists $c \neq 0$ such that $c \cdot N = 0$. If $N \neq 0$, then take any nonzero $\eta = [\frac{a}{x}] \in N$, where x is a system of parameters for R. Since N is F-stable, $F(N) \subseteq N$, thus $cF^e(\eta) = [\frac{ca^{p^e}}{x^{p^e}}] = 0$ for any $e > 0$, contradicting the injectivity of the map $cF^e : H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R)$. Hence $N = 0$.

Conversely, assume that (2) holds. We observe that R is F-injective, otherwise the kernel of the Frobenius $F : H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R)$ would be a nonzero proper F-stable submodule of $H^d_{\mathfrak{m}}(R)$. Now, take $c \in R^{\circ}$ and consider the module

$$T_c = \{\eta \in H^d_{\mathfrak{m}}(R) \mid \; cF^e(\eta) = 0 \;\forall e > 0\}.$$

It is easy to check that T_c is an F-stable submodule of $H^d_{\mathfrak{m}}(R)$. Moreover $cT_c = 0$, therefore $T_c \neq H^d_{\mathfrak{m}}(R)$. Since $H^d_{\mathfrak{m}}(R)$ has no proper nonzero F-stable submodules by assumption, this forces $T_c = 0$. This implies that for any $\eta \in H^d_{\mathfrak{m}}(R)$, there exists $e > 0$ such that $cF^e(\eta) \neq 0$. We define the following family of R-submodules of $H^d_{\mathfrak{m}}(R)$:

$$N_e = \{\eta \in H^d_{\mathfrak{m}}(R) \mid \; cF^e(\eta) = 0\} \text{ for } e > 0$$

By the previous observation, we have $\bigcap_e N_e = T_c = 0$. Moreover, by injectivity of Frobenius on $H^d_{\mathfrak{m}}(R)$ we obtain the chain of inclusions $N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots$. Since $H^d_{\mathfrak{m}}(R)$ is Artinian, these facts imply that there exists an $e_0 > 0$ such that $N_{e_0} = 0$, which is equivalent to say that cF^{e_0} is injective on $H^d_{\mathfrak{m}}(R)$, i.e., R is F-rational. $\square$

Theorem 6.19 (Deformation property). *Let $(R, \mathfrak{m})$ be a local ring and x a regular element on R. Then*

(1) *If R/xR is Cohen-Macaulay and F-injective, then R is Cohen-Macaulay and F-injective.*
(2) *If R/xR is F-rational, then R is F-rational.*

Proof. (1) It is well-known that R/xR implies R Cohen-Macaulay, so it remains to show that the action of Frobenius on $H^d_{\mathfrak{m}}(R)$ is injective, where $d = \dim R$ as usual. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R & \xrightarrow{\cdot x} & R & \longrightarrow & R/xR & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle x^{p^e-1}F^e} & & \downarrow{\scriptstyle F^e} & & \downarrow{\scriptstyle F^e} & & \\
0 & \longrightarrow & R & \xrightarrow{\cdot x} & R & \longrightarrow & R/xR & \longrightarrow & 0.
\end{array}$$

Since R/xR is Cohen-Macaulay of dimension $d-1$, its only non zero local cohomology module is $H^{d-1}_{\mathfrak{m}}(R/xR)$. So, the previous diagram induces the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{d-1}_{\mathfrak{m}}(R/xR) & \longrightarrow & H^d_{\mathfrak{m}}(R) & \xrightarrow{\cdot x} & H^d_{\mathfrak{m}}(R) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle F^e} & & \downarrow{\scriptstyle x^{p^e-1}F^e} & & \downarrow{\scriptstyle F^e} & & \\
0 & \longrightarrow & H^{d-1}_{\mathfrak{m}}(R/xR) & \longrightarrow & H^d_{\mathfrak{m}}(R) & \xrightarrow{\cdot x} & H^d_{\mathfrak{m}}(R) & \longrightarrow & 0.
\end{array}$$

Assume by contradiction that the middle map $x^{p^e-1}F^e$ is not injective on $H^d_{\mathfrak{m}}(R)$. Then $\ker(x^{p^e-1}F^e)$ is a nonzero submodule of $H^d_{\mathfrak{m}}(R)$ which is Artinian. Therefore $\ker(x^{p^e-1}F^e)$ has nonzero intersection with the socle of $H^d_{\mathfrak{m}}(R)$ which is an essential submodule. Thus, there exists $0 \neq \eta \in H^d_{\mathfrak{m}}(R)$ such that $x^{p^e-1}F^e(\eta) = 0$ and $x \cdot \eta = 0$, thus by exactness of the rows of the diagram η is coming from $H^{d-1}_{\mathfrak{m}}(R/xR)$. By diagram chasing, this yields $F^e(\eta) = 0$ in $H^{d-1}_{\mathfrak{m}}(R/xR)$ contradicting the F-injectivity of R/xR. Therefore, the map $x^{p^e-1}F^e : H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R)$ is injective. This forces the injectivity of F^e on $H^d_{\mathfrak{m}}(R)$, that is R is F-injective.

(2) We take $c \in R^\circ$ and consider the F-stable submodule $T_c = \{\eta \in H^d_{\mathfrak{m}}(R) \mid cF^e(\eta) = 0 \ \forall e > 0\}$. Reasoning as in the proof of Theorem 6.18, it is enough to show that $T_c = 0$. Assume by contradiction that $T_c \neq 0$, then also $\mathrm{Soc}(H^d_{\mathfrak{m}}(R)) \cap T_c \neq 0$ since $H^d_{\mathfrak{m}}(R)$ is Artinian so its socle is an essential submodule. This implies the existence of a nonzero $\eta \in H^d_{\mathfrak{m}}(R)$ such that $cF^e(\eta) = 0$ for all $e > 0$ and $x\eta = 0$. We write $c = x^n c'$, where $c' \notin (x)$ and $n \in \mathbb{N}$. Choose $e_0 > 0$ such that $p^{e_0} - 1 \geqslant n$. We consider the commutative diagram of local cohomology modules as in (1). Since $cF^e(\eta) = 0$ for any $e > 0$, by our choice of e_0 we have also $c'x^{p^{e_0}-1}F^{e_0}(\eta) = 0$. Moreover, since $x\eta = 0$ we know that η comes from $H^{d-1}_{\mathfrak{m}}(R/xR)$. Therefore the commutativity of the diagram yields $c'F^{e_0}(\eta) = c'x^{p^{e_0}-1}F^{e_0}(\eta) = 0$, and so $c'F^e(\eta) = 0$ for all $e \gg 0$. On the other hand, R/xR is a normal domain, since it is F-rational, so $c' \neq 0$ in R/xR, that is $c' \in R^\circ$. Thus, the condition $c'F^e(\eta) = 0$ contradicts the injectivity of the map $c'F^e : H^{d-1}_{\mathfrak{m}}(R/xR) \to H^{d-1}_{\mathfrak{m}}(R/xR)$ for $e \gg 0$, hence the F-rationality of R/xR. Hence, $T_c = 0$ and we are done. $\square$

6.3.4. *Rational singularities.* We conclude this section with a brief informal discussion about the connection of F-rationality with an important geometric definition in singularity theory: the notion of rational singularity. Let X be a normal variety over an algebraically closed field k. A *resolution of singularities* for X is a proper[1] birational map $f : W \to X$ such that W is non-singular. A point $x \in X$ is said to be a *rational singularity* if there exists a resolution of singularities $f : W \to X$ such that $(\mathrm{R}^i f_* \mathcal{O}_W)_x = 0$ for all $i \geqslant 1$, where $\mathrm{R}^i f_*(-)$ denotes the i-th right derived functor of the direct image functor $f_*(-)$ (cf. [Har77, §III.8]). Since this is a local condition, in practice it suffices to compute the higher direct images sheaves when X is affine, and in this case $\mathrm{R}^i f_* \mathcal{O}_W$ is the sheaf associated to the module $H^i(W, \mathcal{O}_W)$. In the case that $X = \mathrm{Spec}(R)$ is a surface, with $(R, \mathfrak{m})$ local normal domain, this condition can be further rephrased in a number of ways. For example, there exists among all resolutions of X a *minimal resolution* $\pi : \widetilde{X} \to X$ such that any other resolution factors through $(\widetilde{X}, \pi)$. Then, the origin $x = \{\mathfrak{m}\} \in X$ is a rational singularity if and only if $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 0$, that is the geometric genus of X is 0, or, equivalently, the arithmetic genus of $\widetilde{X}$ and X is the same.

Resolution of singularities are known to exists over fields of characteristic 0 by the work of Hironaka or if the dimension of X is at most 2, but the problem of their existence is still open for higher dimension in positive characteristic. For this reason, Lipman and Teissier introduced the notion of *pseudo-rationality*, which coincides with rationality for rings that are localization of affine domains over fields of characteristic 0. The definition of pseudo-rationality is quite technical and we do not present it here. We limit ourselves to mention that using the characterization of F-rationality of Theorem 6.18, Karen Smith proved that F-finite (and more generally, excellent) rings which are F-rational are pseudo-rational. We refer to [Smi97] for the definition of pseudo-rationality and the proof of this result.

Finally, we consider the following situation. Let $R = k[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ be an affine algebra over a field k of characteristic 0 such that the polynomials $f_1, \ldots, f_m$ have coefficients in $\mathbb{Z}$. Then $\mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ is a free $\mathbb{Z}$-module and we can consider its reduction modulo p for any p prime: $R_p = \mathbb{Z}/p\mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. We say that the ring R has *F-rational type* if R_p is F-rational for all but finitely many prime numbers p. If X is a scheme of finite type over k, and $x \in X$ is a closed point, we say that x has *F-rational type* if x has an open affine neighbourhood defined by a ring of F-rational type. The scheme X has *F-rational type* if every point x of X has F-rational type.

Theorem 6.20 (Smith and Hara)**.** *Let X be a scheme of finite type over an algebraically closed field of characteristic 0. Then X has F-rational type if and only if X has rational singularities.*

[1] A morphism of varieties $\phi : W \to X$ is proper if for every valuation ring V with morphism $\alpha : \mathrm{Spec}(V) \to X$, there is a unique morphism $\beta : \mathrm{Spec}(V) \to W$ such that $\phi \circ \beta = \alpha$.

7. Further relations between F-singularities

So far, given an F-finite ring we have proved the following implications:

$$\begin{tikzcd}
\text{regular} \arrow[r, Rightarrow] & \text{strongly F-regular} \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \text{F-regular} \arrow[r, Rightarrow] & \text{weakly F-regular} \arrow[r, Rightarrow] & \text{F-rational} \arrow[d, Rightarrow] \\
 & & \text{F-split} \arrow[r, Leftrightarrow] & \text{F-pure} \arrow[r, Rightarrow] & \text{F-injective}
\end{tikzcd}$$

To complete the picture of the known implications which always hold between F-singularities we need to prove that weakly F-regular rings are F-split. In order to do so, we need some preliminary results.

When $(R, \mathfrak{m}, k)$ is local, there is a useful criterion to verify whether a map is pure. Recall that the injective hull $E := E_R(k)$ of the residue field k is an injective R-module such that $k \subseteq E$ is an essential extension. The latter means that for every non-zero submodule $N \subseteq E$ one has $N \cap k \neq 0$. The injective hull E exists, and it is unique up to isomorphism.

Remark 7.1. For convenience of the reader, we recall some of the equivalent definitions of a Gorenstein ring. For a proof, see the standard references of this course (e.g., [BH93] or [BS13]). A local ring $(R, \mathfrak{m}, k)$ of dimension d is Gorenstein if R is Cohen-Macaulay and one of the equivalent conditions holds:

(1) $E \cong H^d_{\mathfrak{m}}(R)$;
(2) For some (equivalently, all) system of parameters $(x_1, \ldots, x_d)$, the Artinian ring $\overline{R} = R/(x_1, \ldots, x_d)$ is such that $\operatorname{soc}(\overline{R}) = 0 :_{\overline{R}} \overline{\mathfrak{m}}$ is a 1-dimensional k-vector space;
(3) For some (equivalently, all) system of parameters $(x_1, \ldots, x_d)$, the Artinian ring $\overline{R} = R/(x_1, \ldots, x_d)$ is injective as a module over itself.

Remark 7.2. If R is an Artinian Gorenstein ring and M is any R-module, then any injective map $R \to M$ splits. This can be seen for instance recalling that R is injective as a module over itself, and therefore one obtains a splitting of the inclusion as follows:

$$\begin{tikzcd}
 & R & \\
0 \arrow[r] & R \arrow[u, "\mathrm{id}_R"] \arrow[r] & M \arrow[ul, dashed]
\end{tikzcd}$$

Moreover, since when $(R, \mathfrak{m})$ is Artinian the extension $\operatorname{soc}(R) \subseteq R$ is always essential, when R is Gorenstein the map $f : R \to M$ is injective (hence split) if and only if $f(\delta) \neq 0$ for any generator δ of $\operatorname{soc}(R)$. Note that, in these assumptions, $R \cong E$. More generally, if $f : E \to M$ is a map of R-modules, and $\operatorname{soc}(E) = \langle u \rangle$, then f is split if and only if it is injective, if and only if $f(u) \neq 0$.

We now recall some facts that will be very useful in the rest of this section.

Proposition 7.3. *Let $(R, \mathfrak{m}, k)$ be a local ring, with injective hull of the residue field $k \subseteq E$.*

(1) *The R-module E is Artinian (not Noetherian, unless* $\dim(R) = 0$*).*
(2) *If M is an Artinian R-module, there is an injection $M \hookrightarrow E^{\oplus t}$ for some integer $t > 0$, called the type of M.*
(3) *For any $\alpha \in E$ there exists a positive integer $n = n(\alpha)$ such that $\mathfrak{m}^n \alpha = 0$. Equivalently, $H^0_{\mathfrak{m}}(E) = E$.*

(4) *If R is an excellent reduced ring (e.g. complete or F-finite and reduced), or has depth at least two, then R is approximately Gorenstein, that is, there exists a sequence of nested ideals $\{I_t\}_{t\geqslant 1}$, cofinal withe the powers of the maximal ideal, such that R/I_t is Artinian Gorenstein for all $t \geqslant 1$. Moreover, $E = \lim_{t\to\infty} R/I_t$.*

The first three claims are standard facts about the injective hull of the residue field; for instance, see [BH93]. The first claim of (4) can be found in Hochster's work on purity VS cyclic purity [Hoc77]. For the last claim, observe that if R is approximately Gorenstein with respect to a sequence $\{I_t\}$, then we may assume without loss of generality that $I_{t+1} \subseteq I_t$ for every t. Observe that, since I_t defines a Gorenstein Artinian ring, one has that $\operatorname{Hom}_R(R/I_t, E_R(k)) \cong E_{R/I_t}(k) \cong R/I_t$. In particular, since the powers $\{I_t\}$ are cofinal with the maximal ideal, the inclusions $I_{t+1} \subseteq I_t$ and the Ext-definition of local cohomology yield

$$E = H^0_{\mathfrak{m}}(E) \cong \lim_{t\to\infty} \operatorname{Hom}_R(R/I_t, E) \cong \lim_{t\to\infty} R/I_t.$$

The following result will be crucial in the rest of the section.

Proposition 7.4. *Let $f : (R, \mathfrak{m}) \to S$ be a ring map, and assume that R is approximately Gorenstein with respect to a family of ideals $\{I_t\}$. The following are equivalent:*

(1) *f is pure.*
(2) *$f_{R/I} : R/I \to S/IS$ is injective for all ideals $I \subseteq R$.*
(3) *$f_t : R/I_t \to S/I_tS$ is injective for all $t \gg 0$.*
(4) *$f_E : E \to S \otimes_R E$ is injective.*
(5) *If u denotes the image of $1 \in k$ inside E, then $f_E(u) \neq 0$.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are clear. The fact that (3) implies (4) follows from the aforementioned fact that $E \cong \lim\limits_{t\to\infty} R/I_t$. Clearly (4) implies (5). Assume (5), and assume by contradiction that f is not pure. Since tensor products commute with direct limits, and every R-module is a direct limit of finitely generated R-modules, f is pure if and only if $f_M : M \to S \otimes_R M$ is injective for every finitely generated R-module M. Let M be a finitely generated R-module, and assume that $f_M(\alpha) = 0$ for some $\alpha \in M$. Because M is finitely generated, there exists $n \in \mathbb{N}$ such that $\alpha \notin \mathfrak{m}^n M$, since $\bigcap_{n\geqslant 1} \mathfrak{m}^n M = (0)$. Then $\overline{\alpha} \neq 0$ in $M/\mathfrak{m}^n M$, and there is an induced map $f_{M/\mathfrak{m}^n M} : M/\mathfrak{m}^n M \to S \otimes_R M/\mathfrak{m}^n M$. Observe that $f_{M/\mathfrak{m}^n M}(\overline{\alpha})$ is still zero. Summing up, if f is not pure, we can find an Artinian module M such that f_M is not injective. Since M is Artinian, there is an injection $\iota : M \hookrightarrow E^{\oplus t}$, which induces a commutative square

$$\begin{tikzcd}
E^{\oplus t} \arrow[r, "f_{E^{\oplus t}}"] & S \otimes_R E^{\oplus t} \\
M \arrow[u, "\iota"] \arrow[r, "f_M"] & S \otimes_R M \arrow[u, "\mathrm{id}_S \otimes \iota"']
\end{tikzcd}$$

By assumption, $f_E(u) \neq 0$, and since u generates $\operatorname{soc}(E)$ by previous considerations we have that f_E is injective. Thus, $(f_E)^{\oplus t} = f_{E^{\oplus t}}$ is injective. Chasing the diagram, this gives that f_M is injective, a contradiction. $\square$

Condition (2) of Proposition 7.4 is sometimes referred to as cyclic purity, since purity is tested only for cyclic modules.

Theorem 7.5. *Let R be a weakly F-regular ring. Then R is F-pure.*

Proof. Since both weak F-regularity and F-purity are local issues at maximal ideals, we may assume that $(R, \mathfrak{m})$ is local and weakly F-regular. Then R is normal by Proposition 3.5. If $\dim(R) \leqslant 1$ then R is regular, and we are done. If $\dim(R) \geqslant 2$, then R satisfies Serre's condition (S_2), and in particular $\operatorname{depth}(R) \geqslant 2$. Therefore R is approximately Gorenstein with respect to some family $\{I_t\}$. Suppose that $R \to F_*(R)$ is not pure. Then there exists $t \gg 0$ such that $R/I_t \to F_*(R)/I_tF_*(R)$ is not injective, that is, there exists $r \in R \smallsetminus I_t$ such that $r \in I_tF_*(R)$. As the latter is equivalent to $r^p \in I_t^{[p]}$, this implies that $r \in I_t^*$ choosing $c = 1$, and thus $r \in I_t$ because R is weakly F-regular; a contradiction. $\square$

Remark 7.6. Given an ideal $I \subseteq R$, the Frobenius closure of I is defined as $I^F = \{x \in R \mid x^q \in I^{[q]} \text{ for all } q = p^e \gg 0\}$. Note that $I \subseteq I^F \subseteq I^*$. Proposition 7.4 shows that if R is excellent, then R is F-pure if and only if $I = I^F$ for all ideals $I \subseteq R$. The forward direction is clear, since the map $\varphi_{R/I} : R/I \to F_*^e(R)/IF_*^e(R)$ is injective for all ideals I and all $e \geqslant 1$ if R is F-pure. Conversely, the fact that $(0) = (0)^F$ implies that R is reduced, and thus approximately Gorenstein. Our assumption guarantees that $\varphi_{R/I} : R/I \to F_*^e(R)/IF_*^e(R)$ is injective for all ideals I and all $e \geqslant 1$, and by Proposition 7.4 we conclude that R is F-pure.

Theorem 7.7. *Let R be a Gorenstein ring. If R is F-injective, then R is F-pure. If R is F-rational and F-finite, then R is strongly F-regular.*

Proof. All issues are local at maximal ideals, therefore we may assume that $(R, \mathfrak{m})$ is a d-dimensional Gorenstein local ring. Let $x_1, \ldots, x_d$ be a full system of parameters, and let $I_t = (x_1^t, \ldots, x_d^t)$. Observe that R is approximately Gorenstein with respect to the family of ideals $\{I_t\}$.

First assume that R is F-injective. Consider the map $\varphi : R \to F_*(R)$ sending $1 \mapsto F_*(1)$. By Proposition 7.4 it suffices to show that $\varphi \otimes R/I_t$ is injective for all $t \gg 0$. Assume by way of contradiction that $\varphi \otimes R/I_t$ is not injective for some $t > 0$. This means that there exists $r \in R \smallsetminus I_t$, and such that $r^p \in I_t^{[p]}$. If we consider the element $\eta = \left[\frac{r}{x^t}\right]$, where $x = x_1 \cdots x_d$, then $\eta \neq 0$ but $F(\eta) = 0$, contradicting our assumption. This shows that R is F-pure.

Now assume that R is F-rational and F-finite. Let $c \in R^\circ$, and let $e > 0$ be such that $cF^e : H_\mathfrak{m}^d(R) \to H_\mathfrak{m}^d(R)$ is injective, which exists by assumption. We claim that the map $\varphi_e : R \to F_*^e(R)$ defined as $1 \mapsto F_*^e(c)$ splits. Note that, since R is F-finite, φ_e splits if and only if it is pure. By way of contradiction, assume that φ_e is not pure, so that $\varphi_e \otimes R/I_t$ is not injective for some $t > 0$. This means that there exists $r \in R \smallsetminus I_t$ such that $cr^q = I_t^{[q]}$ for $q = p^e$. If we let $\eta = \left[\frac{r}{x^t}\right] \in H_\mathfrak{m}^d(R)$, then η is a non-zero element in the kernel of cF^e, which contradicts our assumptions. Therefore φ_e is pure, hence split, and R is strongly F-regular. $\square$

We update the diagram of implications:

$$\begin{tikzcd}
\text{regular} \arrow[r, Rightarrow] & \text{strongly F-regular} \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \text{F-regular} \arrow[r, Rightarrow] & \text{weakly F-regular} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{F-rational} \arrow[d, Rightarrow] \arrow[lll, Rightarrow, bend right, "+\text{Gorenstein}"'] \\
& & \text{F-split} \arrow[r, Leftrightarrow] & \text{F-pure} \arrow[r, Rightarrow] & \text{F-injective} \arrow[l, Rightarrow, bend left, "+\text{Gorenstein}"]
\end{tikzcd}$$

We end the section by discussing why certain arrows in the above diagram are not reversible, in general.

We start with the two most challenging ones, which still constitute one of the biggest open problem in the theory of F-singularities.

Conjecture 7.8. Weak and strong F-regularity are equivalent.

Not much is known about this conjecture (or even about the intermediate converse implications) outside the Gorenstein case. M.P. Murthy showed that weakly F-regular rings of finite type over an uncountable field are F-regular, that F-regular rings essentially of finite type over a field of characteristic $p > 5$ are strongly F-regular up to dimension four [AP19]. Moreover, it is known that weak and strong F-regularity are equivalent if R is $\mathbb{N}$-graded over a field [LS99], or if R is $\mathbb{Q}$-Gorenstein on the punctured spectrum (a finiteness condition on the canonical module as an element of the class group). The full conjecture is known in dimension up to three as a consequence of the previous claim, since 2-dimensional local rings with at worst rational singularities have finite class groups thanks to a result of Lipman.

It is very easy to find an example of an F-pure ring that is not weakly F-regular (or F-rational). In fact, since weakly F-regular and F-rational local rings are normal domains, it suffices to take any Stanley Reisner ring which is not regular. For instance, $R = \mathbb{F}_p[\![x,y]\!]/(xy)$.

Continuing with other F-singularities, there are several examples of strongly F-regular rings which are not regular. For instance, any Veronese subring of a regular local ring is strongly F-regular, since it is a direct summand. However, such rings are typically singular; for an explicit example, take $R = \mathbb{F}_p[s^2, st, t^2] \cong \mathbb{F}_p[x,y,z]/(xz-y^2)$.

The arrows "pointing to F-injectivity" are also not reversible, in general.

Example 7.9 (Fedder, Singh)**.** Let $R = \mathbb{F}_p[\![x,y,z,w]\!]/(xy, xz, y(z-w^2))$. If we let $I = (w^2(x^2-y^4))$ and $\alpha = y^4w^3$, then we claim that $\alpha \notin I$, but $\alpha^p \in I^{[p]}$. In fact, modulo $I^{[p]}$ one has:

$$\alpha^p = y^{4p}w^{3p} = y^{4p}w^{3p-2}z = w^{3p-2}x^{2p}z = 0,$$

where we used that $y^2w = yz$ in R, $3p-2 \geqslant 2p$ and $y^{4p}w^{3p-2} = x^{2p}w^{3p-2}$ modulo $I^{[p]}$. Now let us give degrees $\deg(x) = \deg(z) = 2$ and $\deg(y) = \deg(w) = 1$. Then R is graded, and both α and I are homogeneous. If $\alpha \in I$, then there exist homogeneous elements $A, B, C, D \in S = \mathbb{F}_p[x,y,z,w]$ such that in S one has:

$$y^4w^3 = A(w^2(x^2-y^4)) + B(xy) + C(xz) + D(y(z-w^2)).$$

By looking at degrees, we see that A has to have degree one, and thus $A = A(y,w)$. For this reason, it is easy to see that A must be zero modulo (y), and therefore $A = \lambda y$ for some $\lambda \in \mathbb{F}_p$.

Going modulo (x, z), we get the equality $y^4w^3 = -\lambda y^5w^2 - \overline{D}yw^2$, where $\overline{D}$ is the image of D in $S/(x, z)$. We get that $\overline{D} = y^3w + \lambda y^4$, and therefore $D = y^3w + \lambda y^4 + D'$, where $D' \in (x, z)S$. Substituting, we get a new equality $0 = \lambda x^2yw^2 + B(xy) + C(xz) + D'(y(z - w^2)) + y^4zw + \lambda y^5z$. Dividing by y and regrouping, we can find homogeneous polynomials F, G such that $0 = Fx + G(z - w^2) + y^3zw + \lambda y^4z$. Going modulo (x), in $S/(x) \cong \mathbb{F}_p[y, z, w]$ we get an equality $y^3zw + \lambda y^4z = -\overline{G}(z - w^2)$. Dividing by y^3z we get $w + \lambda y = -\overline{G}(z - w^2)$, which is a contradiction, since the left-hand side has degree one, while $z - w^2$ has degree two. Therefore $\alpha \notin I$.

The above claims show that the map $R/I \to F_*(R)/IF_*(R)$ is not injective, and therefore R is not F-pure. Since R is not a domain, it cannot be F-rational. Finally, note that w is a regular element for R, and $R/(w) \cong \mathbb{F}_p[\![x, y, z]\!]/(xy, xz, yz)$ is F-split. Since R is Cohen-Macaulay, it follows from Theorem 6.19 (1) that R is F-injective.

Finally, it is harder to find an example of an F-rational ring which is not weakly F-regular. Again, such an example cannot be Gorenstein, by what we have shown above. The best known way to obtain such examples come from a geometric construction, masterfully used by K.I. Watanabe to construct the following example.

Example 7.10 (Watanabe)**.** Let R be the localization of $\mathbb{F}_p[t, xt^4, x^{-1}t^4, (x-1)^{-1}t^4]$ at the obvious maximal ideal. Then R is F-rational, but is not even F-pure (hence not weakly F-regular). We do not show here the details, and we refer to [MP21, Example 9.1 and Remark 9.2] for a proof of these facts.

8. Hilbert-Kunz Multiplicity

Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of Krull dimension d, let I be an $\mathfrak{m}$-primary ideal, and let M be a finitely generated R-module. The *Hilbert-Samuel function* of I and M is the numerical function $\mathrm{HS}_R(I, M, -) : \mathbb{N} \to \mathbb{N}$ defined as

$$\mathrm{HS}_R(I, M, n) = \ell_R\left(M/I^{n+1}M\right),$$

where $\ell_R(-)$ denotes the R-module length. For $n \gg 0$, this function takes the shape of a polynomial of degree d in n called *Hilbert-Samuel polynomial.* The leading coefficient of this polynomial is $\frac{e(I,M)}{d!}$, where $e(I, M)$ is an integer called *Hilbert-Samuel multiplicity* (or simply multiplicity) of I and M. For $I = \mathfrak{m}$, $M = R$, we usually write $e(R) = e(\mathfrak{m}, R)$ and call it multiplicity of the ring R. The Hilbert-Samuel function and multiplicity capture many important information about the ring.

If we assume further that R has positive characteristic p, then we can replace ordinary powers of the ideal I by Frobenius powers $I^{[p^e]}$ and study the corresponding lengths $\ell_R\left(M/I^{[p^e]}M\right)$ for increasing values of e. This is the approach developed by Kunz in the '60. He was the first to show that the lengths $\ell_R(R/\mathfrak{m}^{[p^e]})$ encode information about the singularities of the ring. Years later, Monsky resumed Kunz's idea and defined the Hilbert-Kunz function and multiplicity which are the main object of investigation of this chapter.

8.1. Rank of $F^e_*(R)$ and Kunz's Theorem revised.

Let R be a Noetherian ring. We recall that an R-module M is said to have rank r if $M \otimes Q$ is a free Q-module of rank r, where Q is the total ring of fractions of R. If M is finitely presented module, then the following facts are equivalent (see e.g. [BH93, Proposition 1.4.3]):

(1) M has rank r;

(2) M has a free submodule F of rank r such that M/F is a torsion module.

Now, assume that $(R,\mathfrak{m},k)$ is local of positive characteristic p and Krull dimension d. During the proof of Kunz's Theorem 2.7, we saw that if R is regular, then the R-module $F_*(R)$ is free of rank $[F_*k:k]p^d$. Using a similar argument, we can compute the rank of $F^e_*(R)$ also when R is not regular.

Theorem 8.1. *Let $(R,\mathfrak{m},k)$ be an F-finite local domain of dimension d. Then for each $e\in\mathbb{N}$ we have that* $\operatorname{rank}_R(F^e_*(R))=[F^e_*k:k]p^{de}$.

Proof. Assume first that R is complete. By Cohen's Structure Theorem there exists a power series subring $A=k[\![x_1,\dots,x_d]\!]\subseteq R$ such that R is a finitely generated A-module. Then we have a commutative diagram of local domains:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & R \arrow[d] \\ F^e_*(A) \arrow[r] & F^e_*(R) \end{tikzcd}$$

which implies

$$\operatorname{rank}_A(F^e_*(R))=\operatorname{rank}_R(F^e_*(R))\cdot\operatorname{rank}_A(R)=\operatorname{rank}_{F^e_*(A)}(F^e_*(R))\cdot\operatorname{rank}_A(F^e_*(A)).$$

Now, the local extension $A\to R$ is isomorphic to $F^e_*(A)\to F^e_*(R)$, therefore $\operatorname{rank}_A(R)=\operatorname{rank}_{F^e_*(A)}(F^e_*(R))$. Thus, we obtain $\operatorname{rank}_R(F^e_*(R))=\operatorname{rank}_A(F^e_*(A))$. As we saw in the proof of Theorem 2.7, $F^e_*(A)$ is a free A-module of rank $\operatorname{rank}_A(F^e_*(A))=[F^e_*k:k]p^{de}$. A basis as A-module is given by

$$\left\{F^e_*(\lambda x_1^{i_1}\cdots x_d^{i_d}) \mid 0\leqslant i_j<p^e \text{ and } \{F^e_*\lambda\} \text{ is a free basis of } F^e_*k \text{ over } k\right\}.$$

Finally, suppose that R is not necessarily complete. Let P be a minimal prime of the completion $\widehat{R}$ such that $d=\dim(R)=\dim(\widehat{R}/P)$. Let K be the fraction field of R and L the fraction field of $\widehat{R}/P$. Since P is a minimal prime of $\widehat{R}$ and $\widehat{R}$ is reduced by Lemma 4.7, we have in fact that $L=\widehat{R}_P$. Then, we have the following chain of isomorphisms

$$F^e_*(L)\cong(F^e_*(\widehat{R}))_P\cong F^e_*(\widehat{R})\otimes_{\widehat{R}}\widehat{R}_P\cong F^e_*(R)\otimes_R\widehat{R}\otimes_{\widehat{R}}\widehat{R}_P\cong F^e_*(R)\otimes_R\widehat{R}_P\cong F^e_*(K)\otimes_K L.$$

Therefore, we have $F^e_*(L)\cong F^e_*(K)\otimes_K L$, and in particular, $[F^e_*(L):L]=[F^e_*(K):K]$. Thus we obtain

$$\operatorname{rank}_R(F^e_*(R))=[F^e_*(K):K]=[F^e_*(L):L]=\operatorname{rank}_{\widehat{R}/P}(F^e_*(\widehat{R}/P))=[F^e_*k:k]p^{de},$$

where the last equality follows from the first part of the proof since $\widehat{R}/P$ is complete. □

Remark 8.2. Let K be an F-finite field. Every element $F_*(r)$ of the Frobenius push forward $F_*(K)$ satisfies the monic polynomial equation $x^p-r=0$. Therefore the degree of the minimal polynomial of every element of $F_*(K)$ divides p. It follows that $[F_*K:K]=p^\alpha$ for some $\alpha\in\mathbb{N}$ and by iterating the Frobenius map we obtain also $[F^e_*K:K]=p^{e\alpha}$ for every $e\in\mathbb{Z}_+$. Therefore, for an F-finite local domain $(R,\mathfrak{m},k)$ of dimension d, we have $[F_*k:k]=p^\alpha$, where $\alpha=\log_p([F_*k:k])$ is an integer. Thus, we can write the rank of $F^e_*(R)$ in Theorem 8.1 also as $\operatorname{rank}_R(F^e_*(R))=p^{e(d+\alpha)}$.

We collect in the following lemma some useful properties of the length function under local ring extension.

Lemma 8.3. *Let $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism.*

(1) *If M is an R-module of finite length and φ is flat then*
$$\ell_S(S \otimes_R M) = \ell_R(M) \cdot \ell_S(S/\mathfrak{m}S).$$
(2) *If N is an S-module of finite length and $[S/\mathfrak{n} : R/\mathfrak{m}] < \infty$ then*
$$\ell_R(N) = [S/\mathfrak{n} : R/\mathfrak{m}] \cdot \ell_S(N).$$

Remark 8.4. Let R be F-finite and M a finitely generated R-module of finite length, then also $F^e_*(M)$ has finite length. So Lemma 8.3 (2) applied to the Frobenius $R \to F^e_*(R)$ yields
$$\ell_R(F^e_*(M)) = [F^e_*k : k] \cdot \ell_{F^e_*(R)}(F^e_*(M)) = [F^e_*k : k] \cdot \ell_R(M).$$
Moreover, for any $\mathfrak{m}$-primary ideal I we have
$$\begin{aligned}\ell_R(F^e_*(M) \otimes_R R/I) &= \ell_R(F^e_*(M)/IF^e_*(M)) \\ &= \ell_R\left(F^e_*(M/I^{[p^e]}M)\right) \\ &= [F^e_*k : k] \cdot \ell_R\left(M/I^{[p^e]}M\right)\end{aligned}$$
In particular, when $I = \mathfrak{m}$ by Nakayama's Lemma the left hand side of the previous chain of equalities is precisely $\mu_R(F^e_*(M))$, the minimal number of generators of $F^e_*(M)$ as R-module, thus
$$\mu_R(F^e_*(M)) = [F^e_*k : k] \cdot \ell_R\left(M/\mathfrak{m}^{[p^e]}M\right).$$

Remark 8.5. For a local ring $(R, \mathfrak{m}, k)$ the $\mathfrak{m}$-adic completion $R \to \widehat{R}$ is a flat map, so by Lemma 8.3 (1), when we compute the length of an R-module M we can assume without loss of generality that R is complete. Similarly, taking the algebraic closure of the residue field is also a flat map, so we can also assume that k is algebraically closed.

Theorem 8.6 (Kunz)**.** *Let $(R, \mathfrak{m}, k)$ be a local F-finite ring of dimension d. Then*
$$\ell_R\left(R/\mathfrak{m}^{[p^e]}\right) \geqslant p^{de} \quad \forall e \geqslant 0.$$
Moreover, the following facts are equivalent:

(1) *R is regular;*
(2) *$\ell_R\left(R/\mathfrak{m}^{[p^e]}\right) = p^{de}$ for some $e \geqslant 0$;*
(3) *$\ell_R\left(R/\mathfrak{m}^{[p^e]}\right) = p^{de}$ for all $e \geqslant 0$;*
(4) *$F^e_*(R)$ is R-free for some $e \geqslant 0$;*
(5) *$F^e_*(R)$ is R-free for all $e \geqslant 0$.*

Proof. To prove the first statement, observe that going modulo a minimal prime will only potentially decrease $\ell_R\left(R/\mathfrak{m}^{[p^e]}\right)$. Therefore we can assume that R is a domain. Moreover, we may also assume that R is complete and k is algebraically closed thanks to the previous observation. By Theorem 8.1, $F^e_*(R)$ is a finitely generated R-module of rank p^{de}, therefore we have $\mu_R(F^e_*(R)) \geqslant p^{de}$ with equality if and only if $F^e_*(R)$ is free. By Nakayama's Lemma (see Remark 8.4) we obtain
$$\ell_R\left(R/\mathfrak{m}^{[p^e]}\right) = \ell_R(F^e_*(R)/\mathfrak{m}F^e_*(R)) = \mu_R(F^e_*(R)) \geqslant p^{de}$$
with equality if and only if $F^e_*(R)$ is free. This shows the first claim and the equivalences (2) $\Leftrightarrow$ (4) and (3) $\Leftrightarrow$ (5). By Kunz's Theorem 2.7 we have that R is regular if and only if $F^e_*(R)$ is flat for some (equivalently for all) $e \geqslant 0$, but for a finitely generated module over

a local ring being flat and free are equivalent. This shows the remaining equivalences and completes the proof. $\square$

8.2. **Existence of the HK multiplicity.** From now on, let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of prime characteristic $p > 0$, and Krull dimension d.

Definition 8.7. Let I be an $\mathfrak{m}$-primary ideal of R, and let M be a finitely generated R-module. The function

$$\mathrm{HK}_R(I, M, e) = \ell_R\left(M/I^{[p^e]}M\right)$$

is called *Hilbert-Kunz function* of I and M. When $M = R$ we will denote the function also by $\mathrm{HK}_R(I, e)$, and if further $I = \mathfrak{m}$ we denote the Hilbert-Kunz function simply by $\mathrm{HK}_R(e)$. The limit

$$\lim_{e\to\infty} \frac{\mathrm{HK}_R(I, M, e)}{p^{de}}$$

is called *Hilbert-Kunz multiplity of I and M* and denoted by $\mathrm{e}_{\mathrm{HK}}(I, M)$. We set also $\mathrm{e}_{\mathrm{HK}}(I) = \mathrm{e}_{\mathrm{HK}}(I, R)$ and $\mathrm{e}_{\mathrm{HK}}(R) = \mathrm{e}_{\mathrm{HK}}(\mathfrak{m})$, the latter is also called *Hilbert-Kunz multiplicity of R*.

It is not clear from the definition that the limit defining the Hilbert-Kunz multiplicity always exists. In fact, although this function was first studied by Kunz at the end of the 60's, the existence of the limit was proved only later by Monsky in 1983. The rest of this section is devoted to the proof of the the existence of the Hilbert-Kunz multiplicity. We will follow Monksy's original path with few adaptations.

Lemma 8.8 (Lech's Formula)**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring of Krull dimension d, let $x_1, \ldots, x_d$ be a system of parameters generating an ideal J, and let M be a finitely generated R-module. Then*

$$\lim_{\min\{a_j\}\to\infty} \frac{\ell_R\left(M/(x_1^{a_1}, \cdots, x_d^{a_d})M\right)}{a_1 \cdots a_d} = e(J, M),$$

where $e(J, M)$ denotes the Hilbert-Samuel multiplicity of J over M.

Proof. If M is maximal Cohen-Macaulay the formula follows easily from the fact that $e(I, M) = \ell_R(M/IM)$ for any $\mathfrak{m}$-primary ideal I. A proof of the general case can be done by induction on d. We refer the interested reader to [HS06, Theorem 11.2.10] for details. $\square$

We will often use the following standard notation.

Notation 8.9. For two functions $f, g : \mathbb{N} \to \mathbb{R}$ we write $f(n) = O(g(n))$ if there exists a constant $C \geqslant 0$ such that $|f(n)| \leqslant C \cdot g(n)$ for all $n \gg 0$.

Lemma 8.10. *Let I be an $\mathfrak{m}$-primary ideal of R, and let M be a finitely generated R-module, then*

$$\frac{e(I, M)}{d!} \leqslant \liminf_{e\to\infty} \frac{\mathrm{HK}_R(I, M, e)}{p^{de}} \leqslant \limsup_{e\to\infty} \frac{\mathrm{HK}_R(I, M, e)}{p^{de}} \leqslant e(I, M).$$

Proof. Since we are considering lengths we can assume without loss of generality that the ring R is complete and the residue field k is algebraically closed. Let $J \subseteq I$ be a minimal reduction. Since I is $\mathfrak{m}$-primary, J is generated by a system of parameters $x_1, \ldots, x_d$. We have inclusions $J^{[p^e]} \subseteq I^{[p^e]} \subseteq I^{p^e}$, which imply inequalities

$$\ell_R\left(M/J^{[p^e]}M\right) \geqslant \ell_R\left(M/I^{[p^e]}M\right) \geqslant \ell_R\left(M/I^{p^e}M\right),$$

where $\ell_R\left(M/I^{[p^e]}M\right) = \mathrm{HK}_R(I,M,e)$ and $\ell_R\left(M/I^{p^e}M\right)$ is the Hilbert-Samuel function of I and M. We recall that $\ell_R\left(M/I^{p^e}M\right) \to \frac{e(I,M)}{d!}p^{de} + O(p^{(d-1)e})$. So dividing the previous inequalities by p^{de} and letting $e \to \infty$ we obtain immediately $\frac{e(I,M)}{d!} \leqslant \liminf_{e\to\infty} \dfrac{\mathrm{HK}_R(I,M,e)}{p^{de}}$. Moreover, by Lech's Formula (Lemma 8.8) we have $\frac{\ell_R\left(M/J^{[p^e]}M\right)}{p^{de}} \to e(J,M) = e(I,M)$ since J is a minimal reduction of I. This gives the other inequality and completes the proof. $\square$

The previous lemma has some immediate consequences. First, it gives an upper bound for the Hilbert-Kunz function, and then it gives the existence of the Hilbert-Kunz multiplicity in dimension one. Also, observe that as consequence of Lech's Formula, for ideals generated by a full system of parameters, the Hilbert-Kunz multiplicity coincides with the Hilbert-Samuel multiplicity. We state these facts as separate corollaries.

Corollary 8.11. *Let I be an $\mathfrak{m}$-primary ideal of R, and let M be a finitely generated R-module, then there exists a positive constant $C = C(I,M) \in \mathbb{R}_+$ such that*

$$\mathrm{HK}_R(I,M,e) \leqslant C \cdot p^{e \dim M}$$

for each $e \in \mathbb{N}$. In particular, if $\dim M < \dim R$ then $\mathrm{e}_{\mathrm{HK}}(I,M) = 0$.

Proof. If $\dim M = \dim R = d$, the conclusion follows directly from the statement of Lemma 8.10. For the case $\dim M < d$, we can reason as follows. Take an element $f \in \mathrm{Ann}_R(M)$ such that $\dim R/f < d$ and let $A = R/f$. Since M is finitely generated as R-module it is also finitely generated as A-module. In particular, there exists a surjective map $\psi : A^n \to M$. Tensoring with $R/I^{[p^e]}$ preserve surjection, so we get

$$\mathrm{HK}_R(I,M,e) = \ell_R\left(M/I^{[p^e]}M\right) = \ell_A\left(M/I^{[p^e]}M\right) \leqslant n \cdot \ell_A\left(A/I^{[p^e]}A\right) \leqslant C \cdot p^{e \dim A}$$

where the second equality holds since A and R have the same residue field and the last inequality holds by Lemma 8.10. An induction argument on $\dim M$ concludes the proof. $\square$

Corollary 8.12. *Let $x_1, \ldots, x_d$ be a full system of parameters generating an ideal J, and let M be a finitely generated R-module. Then $\mathrm{e}_{\mathrm{HK}}(J,M) = e(J,M)$.*

Corollary 8.13. *Let $\dim R = 1$ and let I be an $\mathfrak{m}$-primary ideal, then the Hilbert-Kunz multiplicity of I exists and $\mathrm{e}_{\mathrm{HK}}(I) = e(I)$.*

Proof. Simply set $d = 1$ in the statement of Lemma 8.10. $\square$

When R is a one-dimensional local ring, the Hilbert-Kunz function of an $\mathfrak{m}$-primary ideal I takes the shape $\mathrm{HK}_R(I,e) = e(I)p^e + \varphi(e)$, where φ is a bounded function. Monsky proved that φ is periodic. However, determining φ explicitly is not easy in general.

Example 8.14. Consider the quotient ring $R = k[\![x,y]\!]/(x^a - y^b)$, where $a \geqslant b$ are positive integers. Since $\dim R = 1$, by the previous results the Hilbert-Kunz function of R takes the following form

$$\mathrm{HK}_R(e) = \mathrm{e}_{\mathrm{HK}}(\mathfrak{m})p^e + \varphi_{a,b}(e),$$

where $\mathrm{e}_{\mathrm{HK}}(\mathfrak{m}) = e(\mathfrak{m}) = b$ is the Hilbert-Samuel multiplicity of R and the function $\varphi_{a,b}(e)$ is periodic. In order to compute $\varphi_{a,b}(e)$ more explicitly, recall that

$$\mathrm{HK}_R(e) = \dim_k k[\![x,y]\!]/(x^{p^e}, y^{p^e}, x^a - y^b).$$

So we need to count monomials $x^i y^j$ with $0 \leqslant i, j < p^e$ paying attention that the relation $x^a - y^b$ forces to identify some of them. More precisely, we obtain

$$\varphi_{a,b}(e) = \#\{(i,j) \in \mathbb{N}^2 : \ i - r \geqslant (\alpha - \beta)a, \ 0 \leqslant i < a, \ 0 \leqslant j < p^e\},$$

where $j = \beta b + s$, $0 \leqslant s < b$, $p^e = \alpha a + r$, $0 \leqslant r < a$. For example, for $a = b = 5$ and $p = \pm 2 \mod 5$, the Hilbert-Kunz function of $R = k[\![x, y]\!]/(x^5 - y^5)$ has periodic part equal to

$$\varphi_{5,5}(e) = \begin{cases} -4 & \text{for } e \text{ even} \\ -6 & \text{for } e \text{ odd.} \end{cases}$$

If $\dim R > 1$, then the conclusion of Corollary 8.13 does not hold in general. In fact, the Hilbert-Kunz multiplicity is not necessarily an integer as the following example shows.

Example 8.15. We consider the A_2-singularity $R = k[\![x, y, z]\!]/(y^2 - xz)$, where k is a field of characteristic $p > 2$. A direct, but tedious, computation shows that

$$\mathrm{HK}_R(e) = \frac{3}{2}p^{2e} - \frac{1}{2}.$$

Proposition 8.16. *Let I be an $\mathfrak{m}$-primary ideal and let M, N be finitely generated R-modules. We set $W = R \setminus \bigcup_i P_i$, where the union runs over all minimal primes P_i of R with $\dim R/P_i = d$. If $M_W \cong N_W$ then*

$$\Big|\mathrm{HK}_R(I, M, e) - \mathrm{HK}_R(I, N, e)\Big| = O(p^{(d-1)e}).$$

In particular, $\mathrm{e}_{\mathrm{HK}}(I, M) = \mathrm{e}_{\mathrm{HK}}(I, N)$ and if one exists, also the other exists.

Proof. Since $M_W \cong N_W$ then there exists $\varphi : M \to N$ whose cokernel $C = \mathrm{Coker}\varphi$ is annihilated by some $f \in W$. Consider the exact sequence $M \to N \to C \to 0$ and tensor with $R/I^{[p^e]}$. We obtain

$$M/I^{[p^e]}M \to N/I^{[p^e]}N \to C/I^{[p^e]}C \to 0.$$

Taking lengths we obtain $\mathrm{HK}_R(I, N, e) \leqslant \mathrm{HK}_R(I, M, e) + \mathrm{HK}_R(I, C, e)$. Now, we observe that $\dim C < d$ since $fC = 0$ for $f \in W$, i.e., $\mathrm{Ann}_R(C)$ contains a regular element. Therefore, by Corollary 8.11 $\mathrm{HK}_R(I, C, e) = O(p^{(d-1)e})$. Interchanging the role of M and N gives the desired conclusion. $\square$

Proposition 8.17. *Let I be an $\mathfrak{m}$-primary ideal, and let $0 \to N \to M \to L \to 0$ be a short exact sequence of finitely generated R-modules. Then we have*

$$\mathrm{HK}_R(I, M, e) = \mathrm{HK}_R(I, N, e) + \mathrm{HK}_R(I, L, e) + O(p^{(d-1)e}).$$

In particular, we have

$$\mathrm{e}_{\mathrm{HK}}(I, M) = \mathrm{e}_{\mathrm{HK}}(I, N) + \mathrm{e}_{\mathrm{HK}}(I, L),$$

provided the limit defining $\mathrm{e}_{\mathrm{HK}}(-)$ exist.

Proof. 1) Assume first that R is reduced. Then if P is a minimal prime of R, R_P is a field, thus $M_P \cong N_P \oplus L_P$ and the claim follows from Proposition 8.16.

2) If R is not reduced, choose $e' \geqslant 0$ such that $\left(\sqrt{(0)}\right)^{[p^{e'}]} = 0$ and consider the sequence

$0 \to N \to M \to L \to 0$ as a sequence of $R^{p^{e'}}$-modules. We apply *1)* to this sequence, the ideal $I^{[p^{e'}]} \cap R^{p^{e'}}$, and the reduced ring $R^{p^{e'}}$. This yields

$$\mathrm{HK}_R(I, M, e+e') = \mathrm{HK}_R(I, N, e+e') + \mathrm{HK}_R(I, L, e+e') + O(p^{(d-1)e}).$$

On the other hand, $O(p^{(d-1)e}) = O(p^{(d-1)(e+e')})$. So the proposition is proved. $\square$

Theorem 8.18 (Monsky)**.** *Let I be an $\mathfrak{m}$-primary ideal and M a finitely generated R-module. Then the Hilbert-Kunz multiplicity $\mathrm{e}_{\mathrm{HK}}(I, M)$ exists. In particular, we have*

$$\mathrm{HK}_R(I, M, e) = \mathrm{e}_{\mathrm{HK}}(I, M)p^{de} + O(p^{(d-1)e}).$$

Proof. As already observed, we can assume R complete and k algebraically closed without loss of generality. We take a filtration $0 \subseteq M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ where $M_{i+1}/M_i \cong R/P_i$ with $P_i \in \mathrm{Spec}(R)$. Therefore, by applying Proposition 8.17 we can reduce to the case where $M = R/P$ with $P \in \mathrm{Spec}(R)$. So without loss of generality, we assume that $M = R$ domain. Moreover, since R is complete with algebraically closed residue field it is F-finite (see Remark 4.5). So, by Theorem 8.1 the module $F_*(R)$ is finitely generated and torsion-free of rank p^d. In particular, we have short exact sequences

$$0 \to R^{\oplus p^d} \to F_*(R) \to C_1 \to 0$$

$$0 \to F_*(R) \to R^{\oplus p^d} \to C_2 \to 0$$

with $\dim(C_1), \dim(C_2) < d$. Applying Proposition 8.17 to these sequences yields

$$\left| \mathrm{HK}_R(I, F_*(R), e) - \mathrm{HK}_R(I, R^{\oplus p^d}, e) \right| < D \cdot p^{(d-1)e}$$

for some constant $D \geqslant 0$. Now, observe that $\mathrm{HK}_R(I, F_*(R), e) = \ell_R\left(F_*(R)/I^{[p^e]}F_*(R)\right) = \ell_R(R/I^{[p^{e+1}]})$ and $\mathrm{HK}_R(I, R^{\oplus p^d}, e) = \ell_R\left(R/I^{[p^e]}\right)p^d$ since $R^{\oplus p^d}$ is free of rank p^d. We set $c_e = \ell_R\left(R/I^{[p^e]}\right)p^{-de}$. From the previous inequality we get $|c_{e+1} - c_e| < \frac{D}{p^{e+1}}$. This shows that c_e is a Cauchy sequence. In fact, for any $\varepsilon > 0$, choose $N > 0$ such that $\frac{D}{p^{N+1}} < \varepsilon$. Then for any $e + e' > e > N$ we have

$$\begin{aligned}
|c_{e+e'} - c_e| &= |c_{e+e'} - c_{e+e'-1} + c_{e+e'-1} - \cdots + c_{e+1} - c_e| \\
&\leqslant \frac{D}{p^{e+e'+1}} + \frac{D}{p^{e+e'}} + \cdots + \frac{D}{p^{e+1}} \\
&= \frac{D}{p^{N+1}} \left(\sum_{i=e-N}^{e+e'-N} \frac{1}{p^i} \right) \leqslant \frac{D}{p^{N+1}}\varepsilon.
\end{aligned}$$

Therefore, $c_e = \frac{\mathrm{HK}_R(I,e)}{p^{de}}$ converges to a real number $\mathrm{e}_{\mathrm{HK}}(I)$. Observe that for every e, e' we have that $|c_{e+e'} - c_e| < \frac{D}{p^e}$. By multiplying this relation by p^{ed} and letting $e' \to \infty$ we get that $|\,\mathrm{e}_{\mathrm{HK}}(I)p^{ed} - \mathrm{HK}_R(I, R, e)| \leqslant Dp^{(d-1)e}$, which also gives the last claim. $\square$

Proposition 8.19. *Let I be an $\mathfrak{m}$-primary ideal and M a finitely generated R-module. Then*

$$\mathrm{e}_{\mathrm{HK}}(I, M) = \sum_P \mathrm{e}_{\mathrm{HK}}(I, R/P)\ell_{R_P}(M_P),$$

where the sum runs over all minimal primes P of R with $\dim(R/P) = d$.

Proof. By Proposition 8.17 and Theorem 8.18, the Hilbert-Kunz multiplicity is additive on short exact sequences. We take a filtration $0 \subseteq M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ where $M_{i+1}/M_i \cong R/P_i$ with $P_i \in \operatorname{Spec}(R)$. If $\dim(R/P_i) < d$ then $\mathrm{e}_{\mathrm{HK}}(I, R/P_i) = 0$, thus by using additivity of HK multiplicity, we obtain that $\mathrm{e}_{\mathrm{HK}}(I, M)$ is the sum of $\mathrm{e}_{\mathrm{HK}}(I, R/P)$ for those primes with $\dim(R/P) = d$ counted as many times as R/P appears as one of the quotients M_{i+1}/M_i. We can count this by localizing at P. One sees that the filtration reduces to a filtration of M_P, but then all terms collapse except for those with $(M_{i+1}/M_i)_P \cong (R/P)_P$, which are exactly as many as $\ell_{R_P}(M_P)$. $\square$

Corollary 8.20. *Let $(R, \mathfrak{m})$ be a Noetherian local domain, let I be an $\mathfrak{m}$-primary ideal and M a finitely generated R-module. Then*

$$\mathrm{e}_{\mathrm{HK}}(I, M) = \mathrm{e}_{\mathrm{HK}}(I, R) \cdot \operatorname{rank}_R(M).$$

Proof. We set $r = \operatorname{rank}_R(M)$ and $W = R \setminus \{0\}$. We recall that $R_W = K$ is the fraction field of R. Then we have $M_W \cong K^{\oplus r} \cong (R^{\oplus r})_W$. Therefore, Proposition 8.16 yields $\mathrm{e}_{\mathrm{HK}}(I, M) = \mathrm{e}_{\mathrm{HK}}(I, R^{\oplus r}) = r\, \mathrm{e}_{\mathrm{HK}}(I, R)$, where the last equality follows from Proposition 8.19. $\square$

We conclude this section with two results that allow us to compute some examples of Hilbert-Kunz multiplicity.

Lemma 8.21. *Let $(R, \mathfrak{m})$ be a regular local ring, and let I be an $\mathfrak{m}$-primary ideal. Then*

$$\mathrm{e}_{\mathrm{HK}}(I, R) = \ell_R(R/I).$$

Proof. Since R is a regular local ring, by Kunz's Theorem 8.6 and Theorem 8.1, $F^e_*(R)$ is free of rank $[F^e_* k : k] p^{de}$. Then, by Remark 8.4 we have

$$\begin{aligned}
\mathrm{HK}_R(I, e) &= \ell_R\left(R/I^{[p^e]}\right) \\
&= \frac{1}{[F^e_* k : k]} \ell_R\left(R/I \otimes_R F^e_*(R)\right) \\
&= \frac{1}{[F^e_* k : k]} \ell_R(R/I) \cdot [F^e_* k : k] p^{de} = \ell_R(R/I) \cdot p^{de}, .
\end{aligned}$$

where the third equality follows from the fact that $R/I \otimes_R F^e_*(R) \cong (R/I)^{\oplus \operatorname{rank} F^e_*(R)}$, since $F^e_*(R)$ is free. Thus, dividing the previous chain of equalities py p^{de} and taking the limit for $e \to \infty$ yields the desired claim. $\square$

Lemma 8.22 (Watanabe–Yoshida)**.** *Let $(R, \mathfrak{m}) \subseteq (S, \mathfrak{n})$ be a local extension of local domains such that S is a finitely generated R-module of rank r and $R/\mathfrak{m} = S/\mathfrak{n}$. Let $I \subseteq R$ be an $\mathfrak{m}$-primary ideal. Then*

$$\mathrm{e}_{\mathrm{HK}}(I, R) = \frac{1}{r}\, \mathrm{e}_{\mathrm{HK}}(IS, S).$$

In particular, if S is regular then $\mathrm{e}_{\mathrm{HK}}(I, R) = \frac{1}{r} \ell_S(S/IS)$.

Proof. Observe that since $R/\mathfrak{m} = S/\mathfrak{n}$, by Lemma 8.3 we have

$$\mathrm{HK}_S(IS, S, e) = \ell_S\left(S/(IS)^{[p^e]}\right) = \ell_S\left(S/I^{[p^e]}S\right) = \ell_R\left(S/I^{[p^e]}S\right) = \mathrm{HK}_R(I, S, e).$$

In other words, the Hilbert-Kunz function over the ring S of the ideal IS and the S-module S coincides with the Hilbert-Kunz function over the ring R of the ideal I and the R-module S. Dividing by p^{de} both sides and taking the limit for $e \to \infty$ yields $\mathrm{e}_{\mathrm{HK}}(IS, S) = \mathrm{e}_{\mathrm{HK}}(I, S) =$

$r\,\mathrm{e}_{\mathrm{HK}}(I,R)$, where the last equality follows from Corollary 8.20. Finally, the case when S is regular follows by combining this with the previous lemma. $\square$

We can use the previous lemmas to compute the Hilbert-Kunz multiplicity of several classes of rings.

Example 8.23. Let $S = k[\![x_1,\ldots,x_d]\!]$, and consider a finite group G acting linearly on S such that $p \nmid |G|$. We denote by $R = S^G$ the corresponding invariant ring as in Examples 3.10 and 5.27. It is well known from Invariant Theory that R is a Cohen-Macaulay normal local domain and S is a finitely generated R-module of rank $|G|$. Thus, by Lemma 8.22 we can compute the Hilbert-Kunz multiplicity of R as $\mathrm{e}_{\mathrm{HK}}(R) = \frac{1}{|G|}\dim_k(S/\mathfrak{m}S)$, where $\mathfrak{m}$ is the maximal ideal of R. Sometimes this latter dimension is easier to compute. For example, if G is a cyclic group of order n acting linearly on S by $x_i \mapsto \xi x_i$ where $\xi \in k$ is a primitive n-th root of unity, then $R = S^G$ is the *n-th Veronese subring* of S. In this case we have

$$\mathrm{e}_{\mathrm{HK}}(R) = \frac{1}{n}\binom{d+n-1}{n-1}.$$

Another case of interest is when $d = 2$ and $G \subsetneq \mathrm{SL}(2,k)$. Then the invariant ring $R = k[\![x,y]\!]^G$ is an *ADE singularity* and $\mathrm{e}_{\mathrm{HK}}(R) = 2 - \frac{1}{|G|}$.

8.3. **HK multiplicity and tight closure.** We recall that a local ring R is called *formally equidimensional* or (*quasi-unmixed*) if the dimension of the completion of R modulo any minimal prime is the same, namely the dimension of R. For formally equidimensional rings, David Rees related the multiplicity of an ideal with its integral closure. Namely, he proved that if $(R,\mathfrak{m})$ is a formally equidimensional local ring, and $I \subseteq J$ are $\mathfrak{m}$-primary ideals, then

$$e(I) = e(J) \iff J \subseteq \overline{I}.$$

In particular, $\overline{I}$ is the unique largest ideal containing I having the same multiplicity as I.

Hochster and Huneke proved that a similar relation holds between Hilbert-Kunz multiplicity and tight closure. To prove this, we need the following result by Aberbach, which roughly speaking says that elements not in tight closures are very far from being in Frobenius powers.

Lemma 8.24 (Aberbach)**.** *Let $(R,\mathfrak{m})$ be an excellent local domain such that the completion is also a domain. Let $N = \lim_{\to_t} R/J_t$ be a direct limit system of cyclic modules. Fix $u \notin 0^*_N$. Then there exists $e_0 > 0$ such that*

$$\bigcup_t \left(J_t^{[p^e]} : u_t^{p^e}\right) \subseteq \mathfrak{m}^{[p^{e-e_0}]}$$

for all $e \gg 0$ (where the sequence $\{u_t\}_t$ represents $u \in N$).

Theorem 8.25 (Hochster–Huneke)**.** *Let $(R,\mathfrak{m},k)$ be an excellent local domain such that $\widehat{R}$ is a domain, and let $I \subseteq J$ be $\mathfrak{m}$-primary ideals. Then*

$$\mathrm{e}_{\mathrm{HK}}(I) = \mathrm{e}_{\mathrm{HK}}(J) \iff J \subseteq I^*.$$

In particular, I^* is the unique largest ideal containing I having the same Hilbert-Kunz multiplicity as I.

Proof. "$\Leftarrow$" By assumption there is an element $c \in R^\circ$ such that $cx^{p^e} \in I^{[p^e]}$ for all $x \in J$ and $e \gg 0$. In other words, c annihilates all modules $J^{[p^e]}/I^{[p^e]}$ for $p^e \gg 0$. We observe that these modules have a bounded number of generators, say t, given by the number of generators of J. In particular, we have a surjective map $(R/(c, I^{[p^e]}))^{\oplus t} \to J^{[p^e]}/I^{[p^e]}$, thus $\ell_R\left(J^{[p^e]}/I^{[p^e]}\right) \leqslant t \cdot \ell_R\left(R/(c, I^{[p^e]})\right)$. However, $\ell_R\left(R/(c, I^{[p^e]})\right) = \mathrm{HK}_R(I, R/c, e) = O(p^{(d-1)e})$ since $\dim R/c = d-1$. Therefore, from the short exact sequence $0 \to I^{[p^e]} \to J^{[p^e]} \to J^{[p^e]}/I^{[p^e]} \to 0$ and additivity of $\ell_R(-)$ we obtain $|\mathrm{HK}_R(J,e) - \mathrm{HK}_R(I,e)| = O(p^{(d-1)e})$, which implies $\mathrm{e}_{\mathrm{HK}}(I) = \mathrm{e}_{\mathrm{HK}}(J)$.
"$\Rightarrow$" First, we recall that the Hilbert-Kunz multiplicity does not change by passing to completion, and similarly for any $\mathfrak{m}$-primary ideal L we have $(\widehat{L})^* = \widehat{L^*}$ since tight closure commutes with respect to localization at maximal ideals by Lemma 3.3. Hence, we may assume that R is complete.

Suppose by contradiction that $J \not\subset I^*$, then there exists $x \in J$ such that $x \notin I^*$. We can also assume without loss of generality that $J = (x, I)$. Since $x \notin I^*$, by Lemma 8.24 there exists a fixed integer $e_0 > 0$ such that for $e \gg 0$ we have $I^{[p^e]} : x^{p^e} \subseteq \mathfrak{m}^{[p^{e-e_0}]}$. Therefore, for $e \gg 0$ we have

$$\begin{aligned}
\mathrm{HK}_R(I,e) - \mathrm{HK}_R(J,e) &= \ell_R\left(R/I^{[p^e]}\right) - \ell_R\left(R/(I^{[p^e]}, x^{p^e})\right) \\
&= \ell_R\left(R/(I^{[p^e]} : x^{p^e})\right) \\
&\geqslant \ell_R\left(R/\mathfrak{m}^{[p^{e-e_0}]}\right) \\
&\geqslant \delta p^{de}
\end{aligned}$$

with $\delta > 0$, since $\ell_R\left(R/\mathfrak{m}^{[p^{e-e_0}]}\right) = \mathrm{HK}_R(\mathfrak{m}, e - e_0)$ is the Hilbert-Kunz function of $\mathfrak{m}$ rescaled by a factor of e_0. In particular, this shows that $\mathrm{e}_{\mathrm{HK}}(I) \neq \mathrm{e}_{\mathrm{HK}}(J)$ giving a contradiction. $\square$

There is another important and well-known similarity between Hilbert-Samuel and Hilbert-Kunz multiplicity. Nagata proved that under mild hypothesis the value 1 of the multiplicity characterizes when the ring is regular. More precisely, if $(R, \mathfrak{m})$ is an unmixed local ring, then $e(R) = 1$ if and only if R is regular. Watanabe and Yoshida [WY00] provided the following Hilbert-Kunz analogue of Nagata's Theorem that we quote here without proof. We recall that a local ring R is said to be *formally unmixed* if its completion $\widehat{R}$ is unmixed.

Theorem 8.26 (Watanabe–Yoshida)**.** *Let $(R, \mathfrak{m})$ be a formally unmixed local ring. Then $\mathrm{e}_{\mathrm{HK}}(R) = 1$ if and only if R is regular.*

Example 8.27. The condition formally unmixed is necessary. In fact, the ring $R = k[\![x, y, z]\!]/(xz, xy)$ is not regular, but $\mathrm{e}_{\mathrm{HK}}(R) = 1$.

We close this chapter with two remarks concerning the comparison between Hilbert-Kunz and Hilbert-Samuel function/multiplicity.

Remark 8.28. The Hilbert-Samuel multiplicity $e(R)$ of a local ring $(R, \mathfrak{m})$ is always a positive integer. This is not true for the Hilbert-Kunz multiplicity already in simple cases (see e.g. Example 8.23). However, for a long time all known examples were rational, so it was thought that $\mathrm{e}_{\mathrm{HK}}(R)$ would always be a rational number. While this is true for some classes of rings, such as two-dimensional graded normal rings or binomial hypersurfaces, in 2013 Brenner

[Bre13] gave an example of a local ring of dimension $\geqslant 3$ with irrational Hilbert-Kunz multiplicity.

Remark 8.29. The Hilbert-Samuel function $\mathrm{HS}_R(n)$ takes the shape of a polynomial in n of degree $d = \dim R$ for $n \gg 0$. Despite having a polynomial leading term of degree d in p^e, in general the Hilbert-Kunz function $\mathrm{HK}_R(e)$ is not polynomial in p^e. This is true only in very special cases. However, one may ask whether there exists at least a "second coefficient" for $\mathrm{HK}_R(e)$, that is if there exists $\beta \in \mathbb{R}$ such that

$$\mathrm{HK}_R(e) = e_{HK}(R)p^{de} + \beta p^{(d-1)e} + O(p^{(d-2)e}).$$

This is known to be true for some large classes of rings. Huneke, McDermott, and Monsky [HMM04] proved that, if R is normal, excellent and with perfect residue field, then this is the case. Chan and Kurano [CK16] proved that the same result holds if one replaces normal with regular in codimension one. Additionally, Brenner [Bre07] showed that, for standard graded normal domains of dimension two over an algebraically closed field, the second coefficient β equals zero.

9. F-signature

In this section we closely follow the approach given in [MP21].

9.1. **F-signature exists.** Let $(R, \mathfrak{m})$ be an F-finite local ring. By Kunz's Theorem R is regular if and only if the R-module $F^e_*(R)$ is free for all/some $e > 0$. When R is not regular, in order to measure its distance from being regular Smith and Van den Bergh considered the free part of the modules $F^e_*(R)$ and its asymptotic behavior when e grows. Later, Huneke and Leuschke reprised this idea and defined a new numerical invariant called *F-signature*.

Definition 9.1. Let $(R, \mathfrak{m})$ be a local ring and M a finitely generated R-module. The *free rank* of M is the unique integer $\mathrm{frk}_R(M) \geqslant 0$ such that M admits a direct sum decomposition

$$M = R^{\oplus \mathrm{frk}_R(M)} \oplus N,$$

where the module N has no free direct summands.

The cancellation property of direct sums over local rings guarantees the existence and the uniqueness of $\mathrm{frk}_R M$. In particular, we observe that the fact that N has no free direct summands is equivalent to requiring that $\phi(N) \neq R$ for all $\phi \in \mathrm{Hom}_R(N, R)$. Moreover, the free rank of M can be seen also as the maximum integer n such that there exists a surjection $M \twoheadrightarrow R^{\oplus n}$. Finally, when the ring R is clear from the context we will omit the subscript and denote the free rank of M simply as $\mathrm{frk}\, M$.

Definition 9.2. Let $(R, \mathfrak{m}, k)$ be an F-finite local domain. The *F-signature* of R is

$$s(R) = \lim_{e\to\infty} \frac{\mathrm{frk}_R\left(F^e_*(R)\right)}{\mathrm{rank}_R\left(F^e_*(R)\right)}.$$

As for the Hilbert-Kunz multiplicity, it is not clear from the definition that the limit defining the F-signature always exists. The existence of the limit was proved first in some special cases, and then in full generality in 2011 by Tucker.

Remark 9.3. By Theorem 8.1 we have $\operatorname{rank}_R(F_*^e(R)) = [F_*^e k : k]p^{de}$, where $d = \dim R$. So one may be tempted to replace the denominator in the limit defining the F-signature with $[F_*^e k : k]p^{de}$ and use this to extend the definition of F-signature also to non domains. On the other hand, we will see in Theorem 9.9 that $s(R) = 0$ whenever R is not strongly F-regular. Thus, we will restrict to the domain case from the beginning.

We need some preliminary results on rank and free rank. First, we observe that for any finitely generated R-module M we have inequalities:

$$\operatorname{frk}_R(M) \leqslant \operatorname{rank}_R(M) \leqslant \mu_R(M), \tag{1}$$

where $\mu_R(M)$ is the minimal number of generators of M. Both inequalities become equalities when M is free and are strict otherwise. In particular, they tell us that the F-signature is a real number between 0 and 1.

Lemma 9.4. *Let $(R, \mathfrak{m}, k)$ be a local ring.*

1. *If M_1 and M_2 are R-modules, then $\operatorname{frk}_R(M_1 \oplus M_2) = \operatorname{frk}_R(M_1) + \operatorname{frk}_R(M_2)$.*
2. *If M is an R-module with $M' \subseteq M$ a submodule and $M'' = M/M'$, then*

$$\operatorname{frk}_R(M'') \leqslant \operatorname{frk}_R(M) \leqslant \operatorname{frk}_R(M') + \mu_R(M'').$$

Proof. (1) If $M_1 = R^{\oplus \operatorname{frk}(M_1)} \oplus N_1$ and $M_2 = R^{\oplus \operatorname{frk}(M_2)} \oplus N_2$, where N_1, N_2 do not have free direct summands, then

$$M_1 \oplus M_2 = R^{\oplus(\operatorname{frk}(M_1)+\operatorname{frk}(M_2))} \oplus (N_1 \oplus N_2).$$

It remains to show that $N_1 \oplus N_2$ has no free direct summands. This can be done by passing to the completion $\widehat{R}$ of R and using the KRSA property of complete local rings. Alternatively, one can reason has follows. If $\phi \in \operatorname{Hom}_R(N_1 \oplus N_2, R)$, then $\phi(N_1), \phi(N_2) \subseteq \mathfrak{m}$ as N_1, N_2 have no free direct summands. It follows that $\phi(N_1 \oplus N_2) = \phi(N_1) + \phi(N_2) \subseteq \mathfrak{m}$ as well, therefore $N_1 \oplus N_2$ has no free direct summands.
(2) To prove the first inequality, observe that any surjection $M'' \to R^{\oplus n}$ induces another surjection $M \to R^{\oplus n}$ by pre-composing with the projection $M \to M/M' = M''$. This yields $\operatorname{frk}_R(M'') \leqslant \operatorname{frk}_R(M)$. We prove the second inequality. We decompose M' and M as follows

$$M = R^{\oplus n} \oplus N \quad \text{and} \quad M' = R^{\oplus n} \oplus N',$$

where the inclusion $M' \subseteq M$ is given by equality on $R^{\oplus n}$ and an inclusion $N' \subseteq N$, and $\phi(N') \subseteq \mathfrak{m}$ for any $\phi \in \operatorname{Hom}_R(N, R)$. In other words, n is the maximal rank of a mutual free direct summand of M and M'. Now, consider a surjection $\psi : N \to R^{\oplus \operatorname{frk}(N)}$. Then we have $\psi(N') \subseteq \mathfrak{m}^{\operatorname{frk}(N)}$. Therefore, ψ induces a surjective map $M'' = M/M' = N/N' \to k^{\operatorname{frk}(N)}$ and hence also $M''/\mathfrak{m}M'' \to k^{\operatorname{frk}(N)}$, which shows $\mu(M'') \geqslant \operatorname{frk}(N)$. Putting everything together, we obtain $\operatorname{frk}(M) = \operatorname{frk}(R^{\oplus n}) + \operatorname{frk}(N) = n + \operatorname{frk}(N) \leqslant \operatorname{frk}(M') + \mu(M'')$. ☐

Theorem 9.5 (Tucker)**.** *Let $(R, \mathfrak{m}, k)$ be an F-finite local domain of dimension d. Then the F-signature of R exists.*

Proof. We set $c_e = \frac{\operatorname{frk}_R(F_*^e(R))}{\operatorname{rank}_R(F_*^e(R))}$ for any $e \geqslant 0$, and observe that $c_e \in [0, 1]$ by (1). We reason as in the proof of Theorem 8.18. By Theorem 8.1 the module $F_*(R)$ is finitely generated and torsion-free of rank $[F_*k : k]p^d = p^{d+\alpha}$, where $\alpha = \log_p[F_*k : k]$. Therefore, we have short exact sequences

$$0 \to R^{\oplus p^{d+\alpha}} \to F_*(R) \to C_1 \to 0$$

$$0 \to F_*(R) \to R^{\oplus p^{d+\alpha}} \to C_2 \to 0$$

with $\dim(C_1), \dim(C_2) < d$. Applying the exact functor $F^e_*(-)$ to the previous sequences yields

$$0 \to (F^e_*(R))^{\oplus p^{d+\alpha}} \to F^{e+1}_*(R) \to F^e_*(C_1) \to 0$$

and

$$0 \to F^{e+1}_*(R) \to (F^e_*(R))^{\oplus p^{d+\alpha}} \to F^e_*(C_2) \to 0.$$

We apply Lemma 9.4 to the previous sequences to get

$$|\operatorname{frk}_R\left(F^{e+1}_*(R)\right) - \operatorname{frk}_R\left(F^e_*(R)\right) \cdot p^{d+\alpha}| \leqslant \max\{\mu_R(F^e_*(C_1)), \mu_R(F^e_*(C_2))\}.$$

By Corollary 8.11 and Remark 8.4 there exists a constant $D \geqslant 0$ such that $\mu_R(F^e_*(C_i)) \leqslant Dp^{e \dim C_i}[F^e_*k : k] \leqslant Dp^{(d+\alpha-1)e}$. Dividing by $p^{(e+1)(d+\alpha)}$ we obtain $|c_{e+1} - c_e| \leqslant \frac{D}{p^{e+1}}$. Reasoning as in the proof of of Theorem 8.18, one can show that the sequence $\{c_e\}_e$ is a Cauchy sequence, which implies the existence of the limit $\lim\limits_{e\to\infty} c_e = s(R) \in \mathbb{R}$, i.e., the F-signature exists. □

9.2. **F-signature and strong F-regularity.** The goal of this section is to prove that for F-finite rings having positive F-signature is equivalent to being strongly F-regular. One implication is easier and we prove it immediately.

Definition 9.6. Let $(R, \mathfrak{m})$ be local and F-finite, and let M be a finitely generated R-module. For each $e > 0$, the modules

$$\begin{aligned} I_e(M) &= \{c \in M \mid R \xrightarrow{\cdot F^e_* c} F^e_*(M) \text{ does not split}\} \\ &= \{c \in M \mid \varphi(F^e_*(c)) \in \mathfrak{m} \text{ for every } \varphi \in \operatorname{Hom}_R(F^e_*(M), R)\} \end{aligned}$$

are called *Frobenius non-splitting submodules of M*. When $M = R$ we will denote them simply by $I_e = I_e(R)$.

We prove that $I_e(M)$ is actually a submodule of M and some of its fundamental properties.

Proposition 9.7. *Let R and M be as above, then the following facts hold.*

1. *For any $e > 0$, $I_e(M)$ is a submodule of M and $\mathfrak{m}^{p^e} M \subseteq I_e(M)$.*
2. *For any $e > 0$, $\operatorname{frk}_R(F^e_*(M)) = \ell_R\left(M/I_e(M)\right)[F^e_*k : k]$.*
3. *$\{I_e(M)\}_{e>0}$ is a descending chain of submodules of M.*
4. *If R is strongly F-regular and M is torsion free, then $\bigcap\limits_{e>0} I_e(M) = \{0\}$.*

Proof. (1) Let $\eta_1, \eta_2 \in I_e(M)$ and $r \in R$. We prove that $r\eta_1 + \eta_2 \in I_e(M)$. This is equivalent to the condition that there is no splitting $\varphi \in \operatorname{Hom}_R(F^e_*(M), R)$ such that $\varphi(F^e_*(r\eta_1 + \eta_2)) = 1$. Assume by contradiction that such a splitting φ exists. This implies that $\varphi(F^e_*(r\eta_1) + \varphi(\eta_2)) = 1$. Since R is local, we must have that either $\varphi(F^e_*(r\eta_1))$ is a unit or $\varphi(F^e_*(\eta_2))$ is a unit. In the second case we get directly that $\eta_2 \notin I_e(M)$. In the first case, since $\varphi(F^e_*(r)-) \in \operatorname{Hom}_R(F^e_*(M), R)$, we again obtain that $\eta_1 \notin I_e(M)$. Either way, we get to the desired contradiction.

(2) We decompose $F^e_*(M) \cong R^{\oplus \operatorname{frk}(F^e_*(M))} \oplus N$, where N does not contain free R-summands. Then we obtain $F^e_*(I_e(M)) \cong \mathfrak{m}^{\operatorname{frk}(F^e_*(M))} \oplus N$. Therefore

$$\ell_R\left(M/I_e(M)\right) = \ell_{F^e_*(R)}\left(F^e_*(M/I_e(M))\right) = \frac{\ell_R(F^e_*(M)/F^e_*(I_e(M)))}{[F^e_*k : k]} = \frac{\operatorname{frk}_R(F^e_*(M))}{[F^e_*k : k]}.$$

(3) We want to show that $I_e(M) \supseteq I_{e+1}(M)$. If $I_e(M) = M$ this is clear, so assume $I_e(M) \neq M$. We fix $\eta \in M \setminus I_e(M)$ and we prove that $\eta \notin I_{e+1}(M)$. Since $\eta \notin I_e(M)$ there exists a splitting $\varphi : F^e_*(M) \to R$ such that $\varphi(F^e_*\eta) = 1$. We observe that R is F-split. In fact, the multiplication map $R \xrightarrow{\cdot\eta} M$ induces $F^e_*(R) \xrightarrow{\cdot F^e_*\eta} F^e_*(M)$, and composing with φ we obtain a splitting of $F^e_*(R)$. In particular, we can choose a splitting $F_*R \xrightarrow{\psi}$ such that $\psi(F_*(1)) = 1$. So we obtain $\psi(F_*\varphi(F^{e+1}\eta)) = 1$ which implies $\eta \notin I_{e+1}(M)$ as claimed.

(4) Since M is torsion-free and finitely generated, there exists an injective map into a finite free R-module $M \hookrightarrow R^{\oplus n}$. We fix a non-zero element $\eta \in M$. By composing the previous inclusion with an appropriate projection into one of the factors of $R^{\oplus n}$ we obtain a map $\varphi : M \to R$ such that $\varphi(\eta) = r \neq 0$. Since R is strongly F-regular there exists an $e > 0$ and $\psi : F^e_*(R) \to R$ such that $\psi(F^e_* r) = 1$. Therefore $\psi(F^e_*\varphi(\eta)) = 1$ which implies that $\eta \notin I_e(M)$.

□

We need the following lemma by Chevalley, for a proof we refer to [MP21, Lemma 10.14].

Lemma 9.8 (Chevalley). *Let $(R, \mathfrak{m}, k)$ be a complete local ring and M a finitely generated R-module. Let $I \subseteq R$ be an $\mathfrak{m}$-primary ideal and $\{M_n\}_{n\in\mathbb{N}}$ be a descending chain of R-submodules of M such that $\bigcap_{n\in\mathbb{N}} M_n = \{0\}$. Then there exists an $n_0 \in \mathbb{N}$ such that $M_{n_0} \subseteq IM$.*

Theorem 9.9 (Aberbach–Leuschke). *Let $(R, \mathfrak{m}, k)$ be an F-finite local ring. Then $s(R) > 0$ if and only if R is strongly F-regular.*

Proof. First we prove that if R is not strongly F-regular, then $s(R) = 0$. We decompose $F^e_*(R) \cong R^{a_e} \oplus M_e$, where $a_e = \operatorname{frk}_R(F^e_*(R))$ and M_e does not contain free summands. By Proposition 9.7 we have that $\operatorname{frk}_R(F^e_*(R)) = \ell_R\left(R/I_e(R)\right)[F^e_*k : k] = \ell_R\left(F^e_*(R/I_e(R))\right)$. Since R is not strongly F-regular, there exists an element $c \in R^\circ$ such that the multiplication map $R \xrightarrow{\cdot F^e_* c} F^e_*(R)$ does not split for all $e > 0$. Observe in particular that $\mathfrak{m}F^e_*(R) + \operatorname{span}_{F^e_*(R)}\{F^e_*c\} \subseteq F^e_*(I_e(R))$ for all $e > 0$. Therefore we obtain

$$\begin{aligned}
\operatorname{frk}_R(F^e_*(R)) &= \ell_R\left(F^e_*(R)/F^e_*(I_e(R))\right) \\
&\leqslant \ell_R\left(F^e_*(R)/\left(\mathfrak{m}F^e_*(R) + \operatorname{span}_{F^e_*(R)}\{F^e_*c\}\right)\right) \\
&= \ell_R\left(F^e_*(R/c) \otimes R/\mathfrak{m}\right) \\
&= [F^e_*k : k] \cdot \operatorname{HK}_R(\mathfrak{m}, R/c, e)
\end{aligned}$$

where the last equality follows from Remark 8.4. By Corollary 8.11, we have $\operatorname{HK}_R(\mathfrak{m}, R/c, e) \leqslant C \cdot p^{(d-1)e}$ for some $C \geqslant 0$ since $\dim R/c = d-1$. In particular, dividing by $[F^e_*k : k]p^{de}$ both sides and taking the limit for $e \to \infty$ we obtain that $s(R) = 0$.

Now we prove the converse. First, observe that we can assume that R is complete, since strong F-regularity, free rank, and rank are preserved under completion. We claim that there exists an $e_0 > 0$ such that for all $e > 0$ we have the inclusion $I_{e+e_0} \subseteq \mathfrak{m}^{[p^e]}$. To prove the claim, we fix $e > 0$ and $r \in R \smallsetminus \mathfrak{m}^{[p^e]}$. Notice that $r \in R \smallsetminus \mathfrak{m}^{[p^e]}$ if and only if $F^e_*r \in F^e_*R \setminus \mathfrak{m}F^e_*R$. For simplicity of notation we set $M = F^e_*R$ in what follows. Since R is complete and Cohen-Macaulay (by Theorem 5.25), there exists a canonical module ω_R. We denote by $(-)^\vee = \operatorname{Hom}_R(-, \omega_R)$ the canonical dual. We consider the following short exact sequence

$$0 \to K \to R^{\oplus n} \to M^\vee \to 0$$

where $R^{\oplus n}$ is a finite free R-module. Since M is maximal Cohen-Macaulay, $M^\vee$ is also maximal Cohen-Macaulay by [BH93, Theorem 3.3.10]. Therefore by the Depth Lemma, K is maximal Cohen-Macaulay as well. We dualize the previous sequence, and use the fact that $(M^\vee)^\vee \cong M$ and $\operatorname{Ext}^1_R(N, \omega_R) = 0$ for all MCM R-modules N to obtain a short exact sequence

$$0 \to M \xrightarrow{\psi} \omega_R^{\oplus n} \to K^\vee \to 0.$$

Now, fix a system of parameters $\underline{x} = x_1, \ldots, x_t$ of R, which is then a regular sequence for R. Consider the $\mathfrak{m}$-primary ideal $I = (\underline{x})$. Since the module $\operatorname{Tor}_1^R(R/I, K^\vee)$ can be computed as the first Koszul homology module $H_1(\underline{x}; K^\vee)$, and $\underline{x}$ is a regular sequence on $K^\vee$, we have that $\operatorname{Tor}_1^R(R/I, K^\vee) = 0$, and the previous short exact sequence yields the short exact sequence

$$0 \to M/IM \xrightarrow{\overline{\psi}} \omega_R^{\oplus n}/I\omega_R^{\oplus n} \to K^\vee/IK^\vee \to 0.$$

Consider the Frobenius non-splitting submodules $I_e(\omega_R)$, by Proposition 9.7 $\bigcap_{e>0} I_e(\omega_R) = (0)$. Thus Chevalley's Lemma implies the existence of an index $e_0 > 0$ such that $I_{e_0}(\omega_R) \subseteq I\omega_R$. Notice that e_0 depends only on R and not on $M = F^e_* R$. Moreover, observe that $I_e(\omega_R^{\oplus n}) = I_e(\omega_R)^{\oplus n}$ as a submodule of $\omega_R^{\oplus n}$. As a consequence of the above containment, we get that $I_{e_0}(\omega_R^{\oplus n}) \subseteq I\omega_R^{\oplus n}$. We now claim that $\psi(F^e_*(r)) \notin I_{e_0}(\omega_R^{\oplus n})$. In fact, if this was not the case, then we would have that $\psi(F^e_*(r)) \in I\omega_R^{\oplus n}$, and since $\overline{\psi}$ is injective we would get that $F^e_*(r) \in IM \subseteq \mathfrak{m}M$. A contradiction. It follows that there exists a splitting $\varphi : F^{e_0}_* \omega_R^{\oplus n} \to R$ such that $\varphi(F^e_* r) = 1$. By restricting φ to $F^{e_0}_*(M) = F^{e+e_0}_*(R) \subseteq F^{e_0}_*(\omega_R^{\oplus n})$ we obtain a splitting $F^{e+e_0}_*(R) \to R$ sending $F^{e+e_0}_* r$ to 1, which is equivalent to saying that $r \in R \smallsetminus I_{e+e_0}$. This proves the claimed inclusion $I_{e+e_0} \subseteq \mathfrak{m}^{[p^e]}$.

To conclude the proof, observe that

$$\begin{aligned} \frac{\operatorname{frk}_R(F^{e+e_0}_* R)}{\operatorname{rank}_R(F^{e+e_0}_* R)} &= \frac{\operatorname{frk}_R(F^{e+e_0}_* R)}{[F^{e+e_0}_* k : k] p^{(e+e_0)d}} \\ &= \frac{1}{p^{(e+e_0)d}} \ell_R\left(R/I_{e+e_0}\right) \\ &\geqslant \frac{1}{p^{(e+e_0)d}} \ell_R\left(R/\mathfrak{m}^{[p^e]}\right) \\ &= \frac{1}{p^{(e+e_0)d}} \operatorname{HK}_R(\mathfrak{m}, e). \end{aligned}$$

Therefore, taking the limit for $e \to \infty$ we obtain $s(R) \geqslant \frac{1}{p^{e_0 d}} > 0$ as desired. $\square$

Thanks to the previous theorem, when looking for examples of F-signature we should restrict to strongly F-regular rings. However, computing explicit examples of F-signature is a difficult task, and only few cases are known. Rings of invariants of Examples 3.10 and 5.27 are among them.

Example 9.10. Let $S = k[\![x_1, \ldots, x_d]\!]$ be a power series ring over an algebraically closed field of characteristic $p > 0$ and let $G \subseteq \operatorname{GL}(d, k)$ be a finite group acting linearly on S such that $p \nmid |G|$. We denote by $R = S^G$ the corresponding invariant ring. We assume further that G does not contain pseudoreflections[2]. This is not really restrictive. In fact, by Chevalley–Shephard–Todd Theorem if G is generated by pseudoreflections then S^G is again a regular local ring. In this setting, using the Auslander correspondence one can convert the computation of the free rank of $F^e_*(R)$ into an analogous problem in representation theory of the group G over k. More precisely, $\operatorname{frk}_R(F^e_*(R))$ is equal to the number of copies of the trivial representation of G appearing into the Frobenius twist representation $F^e_*(S/\mathfrak{m}^{[p^e]})$, where $\mathfrak{m} = (x_1, \ldots, x_d)$. Then, using techniques from representation theory of finite groups, one obtains

$$s(R) = \frac{1}{|G|}.$$

Even more, one can prove that $\operatorname{frk}_R(F^e_*(R))$ is a quasi-polynomial in p^e of degree d, leading coefficient $\frac{1}{|G|}$, and "second coefficient" equal to 0.

9.3. **F-signature and regularity.** The goal of this section is to prove that for an F-finite ring R we have $s(R) = 1$ if and only if R is regular. The approach we will use is the one given by Polstra and Smirnov in [PS19]. One implication follows immediately from Kunz's Theorem. In fact, if R is regular then by Theorem 8.6 $F^e_*(R)$ is free for all $e > 0$, thus $\operatorname{frk}_R F^e_*(R) = \operatorname{rank}_R F^e_*(R)$ for all $e > 0$, and consequently $s(R) = 1$.

In order to prove that $s(R) = 1$ implies that R is regular, we will need some preparatory results. We observe that $s(R) = 1$ implies in particular that R is strongly F-regular by Theorem 9.9, thus we will assume that R is strongly F-regular throughout the rest of the section.

Notation 9.11. Let $N \subseteq M$ be finitely generated R-modules. We denote by N-$\operatorname{rk}(M)$ the maximal number of N-direct summands appearing in all possible direct sum decomposition of M.

Lemma 9.12. *Let $(R, \mathfrak{m})$ be an F-finite and strongly F-regular local ring. Suppose M is a finitely generated R-module such that M-$\operatorname{rk}(F^{e_0}_*(R)) > 0$ for some $e_0 > 0$. Then*

$$\liminf_{e\to\infty} \frac{M\text{-}\operatorname{rk}(F^e_*(R))}{\operatorname{rank}_R(F^e_*(R))} > 0.$$

Proof. Suppose that $F^{e_0}_*(R) \cong M \oplus N$. For any $e > 0$, decompose $F^e_*(R) \cong R^{\oplus a_e} \oplus M_e$ where $a_e = \operatorname{frk}_R(F^e_*(R))$ and M_e does not contain free R-summands. Then we have $F^{e+e_0}_*(R) \cong F^{e_0}_*(R)^{\oplus a_e} \oplus F^{e_0}_*(M_e)$, and thus $M^{\oplus a_e}$ is a direct summand of $F^{e+e_0}_*(R)$. Therefore we have M-$\operatorname{rk}(F^{e+e_0}(R)) \geqslant a_e$, and therefore

$$\liminf_{e\to\infty} \frac{M\text{-}\operatorname{rk}(F^e_*(R))}{\operatorname{rank}_R(F^e_*(R))} \geqslant \liminf_{e\to\infty} \frac{\operatorname{frk}_R(F^{e-e_0}_*(R))}{\operatorname{rank}_R(F^e_*(R))} = \frac{s(R)}{\operatorname{rank}_R(F^{e_0}_*(R))} > 0.$$

□

Proposition 9.13. *Let $(R, \mathfrak{m}, k)$ be an F-finite and strongly F-regular local ring and let $P \subseteq R$ be a prime ideal. Then the following facts are equivalent:*

(1) *$\operatorname{frk}_R(F^e_*(R)) = \operatorname{frk}_{R_P}(F^e_*(R_P))$ for all $e > 0$;*
(2) *$s(R) = s(R_P)$.*

[2]An element $\sigma \in \mathrm{GL}(d, k)$ of finite order is called *pseudoreflection* if the fixed subspace $\{v \in k^d : \sigma(v) = v\}$ has dimension $d - 1$. Equivalently, σ has eigenvalue 1 with multiplicity $d - 1$.

Proof. If $\mathrm{frk}_R(F^e_*(R)) = \mathrm{frk}_{R_P}(F^e_*(R_P))$ for all $e > 0$, then $s(R) = s(R_P)$ since $\mathrm{rank}_R(F^e_*(R)) = \mathrm{rank}_{R_P}(F^e_*(R_P))$. We prove that (2) ⇒ (1). So we assume that $s(R) = s(R_P)$. The inequality $\mathrm{frk}_R(F^e_*(R)) \leqslant \mathrm{frk}_{R_P}(F^e_*(R_P))$ always holds since a free R-summand localizes to a free R_P-summand. Assume by contradiction that $\mathrm{frk}_R(F^{e_0}_*(R)) < \mathrm{frk}_{R_P}(F^{e_0}_*(R_P))$ for some $e_0 > 0$. We write $F^{e_0}_*(R) \cong R^{\oplus a_{e_0}} \oplus M_{e_0}$ with $a_{e_0} = \mathrm{frk}_R(F^{e_0}_*(R))$. Then $(M_{e_0})_P$ must contain a free R_P-summand. For each $e > 0$, consider a direct sum decomposition of the form

$$F^e_*(R) \cong R^{\oplus \,\mathrm{frk}(F^e_*(R))} \oplus M_{e_0}^{\oplus M_{e_0}\text{-}\,\mathrm{rk}(F^e_*(R))} \oplus N_e$$

We localize at P and count free summands, we obtain

$$\mathrm{frk}_{R_P}(F^e_*(R_P)) \geqslant \mathrm{frk}(F^e_*(R)) + M_{e_0}\text{-}\,\mathrm{rank}(F^e_*(R)).$$

This implies

$$\begin{aligned} s(R_P) &= \lim_{e\to\infty} \frac{\mathrm{frk}_{R_P}(F^e_*(R_P))}{\mathrm{rank}_{R_P}(F^e_*(R_P))} \geqslant \lim_{e\to\infty} \frac{\mathrm{frk}_R(F^e_*(R))}{\mathrm{rank}_R(F^e_*(R))} + \liminf_{e\to\infty} \frac{M_{e_0}\text{-}\,\mathrm{rk}(F^e_*(R))}{\mathrm{rank}_R(F^e_*(R))} \\ &= s(R) + \liminf_{e\to\infty} \frac{M_{e_0}\text{-}\,\mathrm{rk}(F^e_*(R))}{\mathrm{rank}_R(F^e_*(R))}. \end{aligned}$$

By Lemma 9.12 we get that $s(R_P) > s(R)$, which gives a contradiction. □

Theorem 9.14. *Let $(R, \mathfrak{m}, k)$ be an F-finite local ring. Then $s(R) = 1$ if and only if R is regular.*

Proof. If R is regular, then $F^e_*(R)$ is free for all $e > 0$ by Kunz's Theorem 8.6, thus $\mathrm{frk}_R(F^e_*(R)) = \mathrm{rank}_R(F^e_*(R))$ which implies $s(R) = 1$. We prove the other implication. Assume $s(R) = 1$, then by Theorem 9.9 R is strongly F-regular, in particular it is a domain. Consider the localization of R at the prime ideal (0) and observe that $R_{(0)} = Q$ is a field, hence a regular ring. So we have $s(Q) = 1 = s(R)$. By Proposition 9.13, we obtain

$$\mathrm{frk}_R(F^e_*(R)) = \mathrm{frk}_Q(F^e_*(Q)) = \dim_Q(F^e_*(Q)) = \mathrm{rank}_R(F^e_*(R))$$

for all $e > 0$. So R is regular by Kunz's Theorem. □

10. Applications

In this final section we present some applications of the F-singularities we have discussed.

10.1. Uniform containments between symbolic and ordinary powers.

Definition 10.1. Let R be a ring, and $I \subsetneq R$ be a radical ideal. The n-th symbolic power of I is defined as

$$I^{(n)} = I^n R_W \cap R,$$

where $W = R \smallsetminus \bigcup_{P \in \mathrm{Min}(I)} P$.

Examples. (1) Let $R = k[x, y, z]$, and $I = (xy, xz, yz)$. Then $I^{(2)} = (x, y)^2 \cap (x, z)^2 \cap (y, z)^2$. Observe that $xyz \in I^{(2)} \smallsetminus I^2$.
(2) Let $R = k[x, y, z]/(x^2 - yz)$, and $Q = (x, y)$. Then $Q^{(2)} = (y)$.
(3) Let $R = \mathbb{F}_p[x_1, x_2, x_3, x_4]$, and Q be the kernel of the $\mathbb{F}_p$-algebra homomorphism from R to $\mathbb{F}_p$ that sends $x_1 \mapsto t^{p^2}$, $x_2 \mapsto t^{p(p+1)}$, $x_3 \mapsto t^{p^2+p+1}$ and $x_4 \mapsto t^{(p+1)^2}$. Then $f = x_1^{p+1} x_2 - x_2^{p+1} - x_1 x_3^p + x_4^p \in Q^{(2)} \smallsetminus Q^2$.

Let us start with some properties of symbolic powers that are well-known and easy to prove. They are here stated for prime ideals, but hold more generally.

Properties 10.2. *Let R be a ring, and $Q \in \operatorname{Spec}(R)$. Then*

(1) *$Q^n \subseteq Q^{(n)}$ for all $n \in \mathbb{N}$.*
(2) *$Q^{(n)}Q^{(m)} \subseteq Q^{(n+m)}$ for all $m, n \in \mathbb{N}$. In particular, $(Q^{(m)})^n \subseteq Q^{(mn)}$.*
(3) *$Q^{(m)} \subseteq Q^{(n)}$ for all $m \geqslant n$.*

Question (Q1)**.** *Given $n \in \mathbb{N}$, is it true that $Q^{(k)} \subseteq Q^n$ for $k \gg 0$? Since $Q^k \subseteq Q^{(k)}$ always holds, in this case $\{Q^n\}$ and $\{Q^{(n)}\}$ describe the same topology (we write $\{Q^n\} \sim \{Q^{(n)}\}$).*

It turns out that the answer is affirmative for regular rings.

Theorem (Swanson)**.** *In the notation and setup considered above, if $\{Q^n\} \sim \{Q^{(n)}\}$, then there exists h (possibly depending on Q) such that $Q^{(hn)} \subseteq Q^n$ for all n.*

Theorem 10.3 (Ein-Lazarsfeld-Smith, Hochster-Huneke)**.** *Let R be a regular ring containing a field, and $Q \in \operatorname{Spec}(R)$ be a prime ideal. Let $h = \max\{1, \operatorname{ht}(Q)\}$. Then $Q^{(hn)} \subseteq Q^n$ for all integers $n \in \mathbb{N}$. In particular, if R has finite Krull dimension d, and we let $H = \max\{1, d-1\}$, then $Q^{(Hn)} \subseteq Q^n$ for all $n \in \mathbb{N}$ and all $Q \in \operatorname{Spec}(R)$.*

Before proving the theorem, we need two basic lemmas.

Lemma 10.4. *Let R be a commutative Noetherian ring, and I, J be two ideals. Then $J \subseteq I$ if and only if $J_\mathfrak{p} \subseteq I_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}_R(R/I)$.*

Proof. The "only if" direction is trivial. For the converse, assume that $J_\mathfrak{p} \subseteq I_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}_R(R/I)$, and let $x \in J$. Consider the ideal $I :_R x$, and the injection

$$0 \longrightarrow \frac{R}{(I :_R x)} \longrightarrow R/I$$

that we have already discussed in previous sections. Because of this, we have that $\operatorname{Ass}_R(R/I :_R x) \subseteq \operatorname{Ass}_R(R/I)$. Let $Q \in \operatorname{Ass}_R(R/I :_R x)$, and observe that $(I :_R x)_Q \cong I_Q :_{R_Q} x = R_Q$, since $Q \in \operatorname{Ass}_R(R/I)$ and $x \in J_Q \subseteq I_Q$ by assumption. It follows that $(R/I :_R x)_Q = 0$, which contradicts the fact that $Q \in \operatorname{Ass}_R(R/I :_R x) \subseteq \operatorname{Supp}_R(R/I :_R x)$. $\square$

Lemma 10.5. *Let R be a regular ring of prime characteristic $p > 0$, and $Q \in \operatorname{Spec}(R)$. Then $\operatorname{Ass}_R(R/Q^{[p]}) = \{Q\}$.*

Proof. Since $\sqrt{Q^{[p]}} = Q$, it is clear that $Q \in \operatorname{Min}(Q^{[p]}) \subseteq \operatorname{Ass}_R(R/Q^{[p]})$. For the converse, let $P \in \operatorname{Ass}_R(R/Q^{[p]})$, so that we can write $P = Q^{[p]} :_R \alpha$ for some $\alpha \in R$. It is clear that $Q \subseteq P$. For the converse, let $r \in P$. Since $\alpha r \in Q^{[p]}$, a fortiori we have $\alpha r^p \in Q^{[p]}$. Equivalently, $\alpha \in Q^{[p]} :_R r^p = (Q :_R r)^{[p]}$, where the last containment follows from the flatness of Frobenius, Theorem 2.7. If $r \notin Q$, then $Q :_R r = Q$, because Q is a prime. It follows that $\alpha \in Q^{[p]}$, hence $P = Q^{[p]} :_R \alpha = R$, a contradiction. Therefore $r \in Q$, and the proof is complete $\square$

We are now ready to prove Theorem 10.3. The generality in which we present it here, as well as its proof, is due to Hochster and Huneke.

Proof of Theorem 10.3. We only show the theorem for the prime characteristic $p > 0$ case. The equal characteristic 0 case is done by reduction to positive characteristic.

For the second claim, observe that if $\operatorname{ht}(Q) = d$, then Q is maximal. Then $Q^{(Hn)} = Q^{Hn} \subseteq Q^n$ for all n. If $\operatorname{ht}(Q) \leqslant d-1$, then $H \geqslant h = \max\{1, \operatorname{ht}(Q)\}$, and the claim follows from the first part.

For the first part, first assume that $\operatorname{ht}(Q) = 0$. Since $R \cong R_1 \times \ldots \times R_t$ is a product of regular domains, Q must be of the form $R_1 \times \ldots \times 0 \times \ldots \times R_t$. Then, as $h = 1$, we have $Q^{(hn)} = Q = Q^n$, and the claim is proved. Now assume that $\operatorname{ht}(Q) > 0$, so that $h = \operatorname{ht}(Q)$.

We start by proving a stronger statement for special values of n, namely, powers of p.

Claim. *For all $q = p^e$, we have $Q^{(qh)} \subseteq Q^{[q]}$.*

Proof of the Claim. Fix $q = p^e$. By Lemma 10.4, in order to show the containment we can localize at the associated primes of $Q^{[q]}$. By Lemma 10.5, we have $\operatorname{Ass}_R(R/Q^{[q]}) = \{Q\}$. Therefore we can show the containment by checking that it holds after localizing at Q. However, $Q^{(hq)}R_Q = (QR_Q)^{hq}$, and since R_Q is regular, its maximal ideal QR_Q is generated by $\dim(R_Q) = \operatorname{ht}(Q)$ elements, say $x_1, \ldots, x_h$. But an element in $(QR_Q)^{hq}$ can be written as $\sum r_{\underline{i}} x_1^{i_1} \cdots x_h^{i_h}$, where $|\underline{i}| = i_1 + \ldots + i_h \geqslant qh$. By the pigeonhole principle, we must have $i_j \geqslant q$ for some j, otherwise $|\underline{i}| < hq$. But then $(QR_Q)^{qh} \subseteq (QR_Q)^{[q]}$, and the claim is proved. $\square$

Since $Q^{[q]} \subseteq Q^q$, the theorem is proved for all $n = p^e$. For the other values, fix $n \in \mathbb{N}$. For $q = p^e > n$, write $q = a_e n + r$, with $0 \leqslant r < n$. We have

$$\left(Q^{(hn)}\right)^{[q]} \subseteq \left(Q^{(hn)}\right)^{a_e n + r} \subseteq \left(Q^{(hn)}\right)^{a_e n} \subseteq \left(Q^{(ha_e n)}\right)^{n}.$$

Choose $0 \neq c \in Q^{hn^2}$. Observe that c is independent of e. We have

$$c\left(Q^{(hn)}\right)^{[q]} \subseteq Q^{hn^2} \cdot \left(Q^{(ha_e n)}\right)^n \subseteq \left(Q^{(hr)}Q^{(ha_e n)}\right)^n \subseteq \left(Q^{h(a_e n + r)}\right)^n = \left(Q^{(hq)}\right)^n \subseteq \left(Q^{[q]}\right)^n = (Q^n)^{[q]}.$$

Since the containment holds for all $q = p^e \gg 0$, we have $Q^{(hn)} \subseteq (Q^n)^* = Q^n$, because regular rings are weakly F-regular by Theorem 2.10. $\square$

10.2. **Direct summands of regular rings are Cohen-Macaulay.** Let $\varphi : R \hookrightarrow S$ be a ring inclusion. We say that the inclusion is split (or that R is a direct summand of S) if there exists an R-module map $\psi : S \to R$ such that $\psi \circ \varphi = \operatorname{id}_R$.

Theorem 10.6 (Hochster-Roberts / Hochster-Huneke / Heitmann-Ma)**.** *Let R be a direct summand of a regular ring S. Then R is Cohen-Macaulay.*

When $(R, \mathfrak{m})$ is local, we can always reduce to the case when S (hence R) is a domain, as a consequence of the following lemma.

Lemma 10.7. *Let $(R, \mathfrak{m})$ be a direct summand of a ring $S \cong S_1 \times \ldots \times S_t$. Then R is a direct summand of S_i for some i.*

Proof. Let $\psi : S \to R$ be the splitting, and let $e_1, \ldots, e_t$ be the idempotents in S that correspond to $1_{S_1}, \ldots, 1_{S_t}$ inside S. Then

$$1 = \psi(1) = \psi(e_1 + \ldots + e_t) = \psi(e_1) + \ldots + \psi(e_t).$$

Since R is local, there exists i such that $\psi(e_i)$ is a unit in R. Consider the natural inclusion and projection $S_i \xrightarrow{\iota} S \xrightarrow{\pi} S_i$. Then $\varphi_i = \pi\varphi$ makes R into a subring of S_i. We then have

$$\begin{array}{ccccccccc} R & \xrightarrow{\varphi_i} & S_i & \xrightarrow{\iota} & S & \xrightarrow{\psi} & R & \xrightarrow{\cdot\psi(e_i)^{-1}} & R \\ 1 & \longrightarrow & 1_{S_i} & \longrightarrow & e_i & \longrightarrow & \psi(e_i) & \longrightarrow & 1 \end{array}$$

so that $\cdot(\psi(e_i))^{-1}\psi \circ \iota : S_i \to R$ gives the desired splitting of φ_i. $\square$

Since regular rings are products of finitely many regular domains, Lemma 10.7 allows indeed to reduce from the case when $(R, \mathfrak{m})$ is a direct summand of a regular ring, to the case when $(R, \mathfrak{m})$ is a direct summand of a regular domain.

Remark 10.8. Observe that, if $(R, \mathfrak{m})$ is a complete local ring that is a direct summand of a regular ring S, then putting together Lemma 10.7, Proposition 3.9, and Theorem 3.7, we get that R is Cohen-Macaulay. This proves Theorem 10.6 in the case when $(R, \mathfrak{m})$ is complete local.

To prove Theorem 10.6 in its full generality, we need to reduce to the case when $(R, \mathfrak{m})$ is complete local. Let $\varphi : R \hookrightarrow S$ be a split ring map, with S regular. Let $\mathfrak{m} \in \operatorname{Max}(R)$, and let $W = \varphi(R \smallsetminus \mathfrak{m})$. Then φ induces an inclusion $R_\mathfrak{m} \to S_W$. Moreover, the original splitting $\psi : S \to R$ induces a map $\psi : S_W \to R_\mathfrak{m}$, that still splits the inclusion $R_\mathfrak{m} \subseteq S_W$. Since localization of a regular ring is regular, and a ring is Cohen-Macaulay if and only if every localization at a maximal ideal is, we may henceforth assume that $(R, \mathfrak{m})$ is local, and it is a direct summand of a regular ring S.

Proposition 10.9. *Let $\varphi : (R, \mathfrak{m}) \to S$ be a pure map, and assume that $S = S_W$ where $W = \varphi(R \smallsetminus \mathfrak{m})$. Then $\widehat{R} \to \widehat{S}$ is pure, where $\widehat{S}$ is the completion of S at the ideal $\mathfrak{m}S$. Moreover, if S is regular, so is $\widehat{S}$.*

Proof. First, observe that $S \to \widehat{S}$ is faithfully flat (it is flat, and maximal ideals in $\widehat{S}$ are maximal ideals of S that contain $\mathfrak{m}S$. Since $S = S_W$ by assumption, these are all the maximal ideals of S, so the map $\operatorname{Spec}(\widehat{S}) \to \operatorname{Spec}(S)$ is surjective; these two conditions give faithful flatness). Then $R \to \widehat{S}$ is pure. Tensoring with E, we then have an injection $E \hookrightarrow \widehat{S} \otimes_R E$, and since $E = E_{\widehat{R}}(k)$, we have $\widehat{R} \otimes_{\widehat{R}} E_{\widehat{R}}(k) \hookrightarrow \widehat{S} \otimes_{\widehat{R}} E_{\widehat{R}}(k)$. By Proposition 7.4, this gives that $\widehat{R} \to \widehat{S}$ is pure.

For the second claim, we note that the completion of $\widehat{S}$ at a maximal ideal Q is isomorphic to the completion of S_Q at the ideal QS_Q, hence regular. It follows that $\widehat{S}$ is regular at every maximal ideal, hence regular. $\square$

Lemma 10.10. *Let $\varphi : (R, \mathfrak{m}) \to S \cong S_1 \times \ldots \times S_t$ be a pure ring map. There exists i such that $(R, \mathfrak{m}) \to S_i$ is pure.*

Proof. By assumption, the image of 1 under φ_E is not zero in $S \otimes_R E \cong \bigoplus(S_i \otimes_R E)$. Therefore, the image $u \in E$ of $1 \in k$ is not zero in $S_i \otimes E$ for some i, and thus $R \to S_i$ is pure by Proposition 7.4. $\square$

We are now ready to prove Theorem 10.6 in full generality.

Proof of Theorem 10.6. Assume that R is a direct summand of a regular ring S. First, we can localize at $\mathfrak{m}$ and $W = \varphi(R \smallsetminus \mathfrak{m})$, and complete by Proposition 10.9. By Lemma 10.10, since regular rings are products of regular domains, we may assume that $(R, \mathfrak{m}) \to S$ is a pure map, with S a regular domain and R complete local. Remark 10.8 now applies, and concludes the proof. $\square$

10.3. **Direct Summand Conjecture (Theorem) in characteristic p.** Let $(R, \mathfrak{m})$ be a complete regular local ring, and let $\iota : R \subseteq S$ be a finite extension. Then R is a direct summand of R, that is, there exists $\psi \in \operatorname{Hom}_R(S, R)$ such that $\psi\iota = \mathrm{id}_R$.

Remark 10.11. Let $P \in \operatorname{Min}(R)$, such that $P \cap R = 0$. Observe that $R \subseteq S/P$ is still a finite extension, and if R is a direct summand of S/P, then $S \to S/P \to R$ gives a splitting of $R \subseteq S$.

Theorem 10.12. *Let $R \subseteq S$ be a finite extension of integral domains. For all ideals $I \subseteq R$ we have $IS \cap R \subseteq I^*$.*

Proof. Let $W = R^\circ$. Since R_W is a field, the inclusion $R_W \subseteq S_W$ splits. Therefore, there exists a map $\varphi \in \operatorname{Hom}_{R_W}(S_W, R_W)$ which splits the inclusion. Since S is a finite R-module, by Remark 5.23 there exists an R-linear map $\psi : S \to R$ and $c \in R^\circ$ such that $\psi(1) = c \neq 0$. Let $x \in I$, so that $x \in IS \cap R$. Raising to the power q, this gives $x^q \in I^{[q]}S$. Applying the map ψ, this gives $\psi(x^q) = x^q\psi(1) = cx^q \in \psi(I^{[q]}S) \subseteq I^{[q]}$. As this happens for all q, and $c \in R^\circ$, we have that $x \in I^*$. $\square$

Remark 10.13. We have seen in Corollary 5.5 that being F-pure and being F-split are equivalent for F-finite rings. Using the same principle (with the same proof), one can show that a finite map $\varphi : R \to S$ is split if and only if it is pure.

Theorem 10.14. *Let R be a regular ring, and let $\varphi : R \hookrightarrow S$ be a finite ring extension. Then R is a direct summand of S.*

Proof. First we reduce to the case in which R is local. Observe that $\varphi : R \to S$ is split if and only if the natural map $\operatorname{Hom}_R(S, R) \to \operatorname{Hom}_R(R, R)$ is surjective (arguing as above). This map is surjective if and only if this is true after localizing at every maximal ideal $\mathfrak{m}$. In turn, since S is a finitely generated R-module, this is equivalent to the map $\operatorname{Hom}_{R_\mathfrak{m}}(S_W, R_\mathfrak{m}) \to \operatorname{Hom}_{R_\mathfrak{m}}(R_\mathfrak{m}, R_\mathfrak{m})$, where $W = \varphi(R \smallsetminus \mathfrak{m})$, being surjective for all $\mathfrak{m} \in \operatorname{Max}(R)$. Finally, this is equivalent to the map $R_\mathfrak{m} \to S_W$ being split for all $\mathfrak{m} \in \operatorname{Max}(R)$, and W as above. Thus, we reduced to the case when R is local. Let $x_1, \ldots, x_d$ be a full system of parameters, and for $t > 0$ let $I_t = (x_1^t, \ldots, x_d^t)$. Since R is regular, hence Gorenstein, R is approximately Gorenstein with respect to the family $\{I_t\}$. By Remark 10.13, it suffices to show that $\varphi : (R, \mathfrak{m}) \to S$ is pure, and by Proposition 7.4 this is equivalent to $\varphi_t : R/I_t \to S/I_tS$ being injective. By Theorem 10.12 we have $I_tS \cap R \subseteq I_t^* = I_t$, since R is regular hence weakly F-regular. But this exactly says that the map $R/I_t \to S/I_tS$ is injective for every $t > 0$, as desired. $\square$

10.4. **Regular local rings of characteristic $p > 0$ are UFDs.** Of course, this is true regardless of the characteristic, but thanks to the flatness of the Frobenius map there is an alternative proof in characteristic $p > 0$.

Theorem 10.15. *Let $(R, \mathfrak{m})$ be a regular local ring of characteristic $p > 0$. Then R is a UFD.*

Proof. We may assume that $d = \dim(R) \geqslant 2$, otherwise the claim is trivial since R is either a field or a DVR. Since R is regular, it is normal (R_1 and S_2). Therefore R is a UFD if and only if every height one prime is principal. Let P be a prime of height one. For all $q = p^e$, we claim that $P^{[q]} = P^{(q)}$. Clearly we have $P^{[q]} \subseteq P^q \subseteq P^{(q)} = (PR_P)^q \cap R = (PR_P)^{[q]} \cap R$, where the last equality follows from the fact that R_P is a DVR, hence PR_P is principal. By Lemmas 10.4 and 10.5, since the only associated prime of $P^{[q]}$ is P we can check the equality after localizing at P, and locally we have $P^{[q]}R_P = ((PR_P)^{[q]} \cap R)R_P$. In particular we obtain that $P^{[q]} = P^q$. If $P = (f_1, \ldots, f_t)$, then it follows that $\mu(P^q) \leqslant t$. This implies that the fiber cone $\mathcal{F}(P) = \bigoplus_{n \geqslant 0} P^n/\mathfrak{m}P^n$ has Krull dimension one (this dimension is called the analytic spread of P). If we let $k = R/\mathfrak{m}$, we can then find an element $x \in P$ and a Noether normalization $k[x] \subseteq \mathcal{F}(P)$. Since this map is finite, there exists $N \geqslant 1$ such that $P^{N+r} = x^r P^N$ for all $r \geqslant 1$. In particular, (x) and P have the same integral closure. Since R is normal, principal ideals are integrally closed, and therefore $\overline{P} = P = (x)$. $\square$